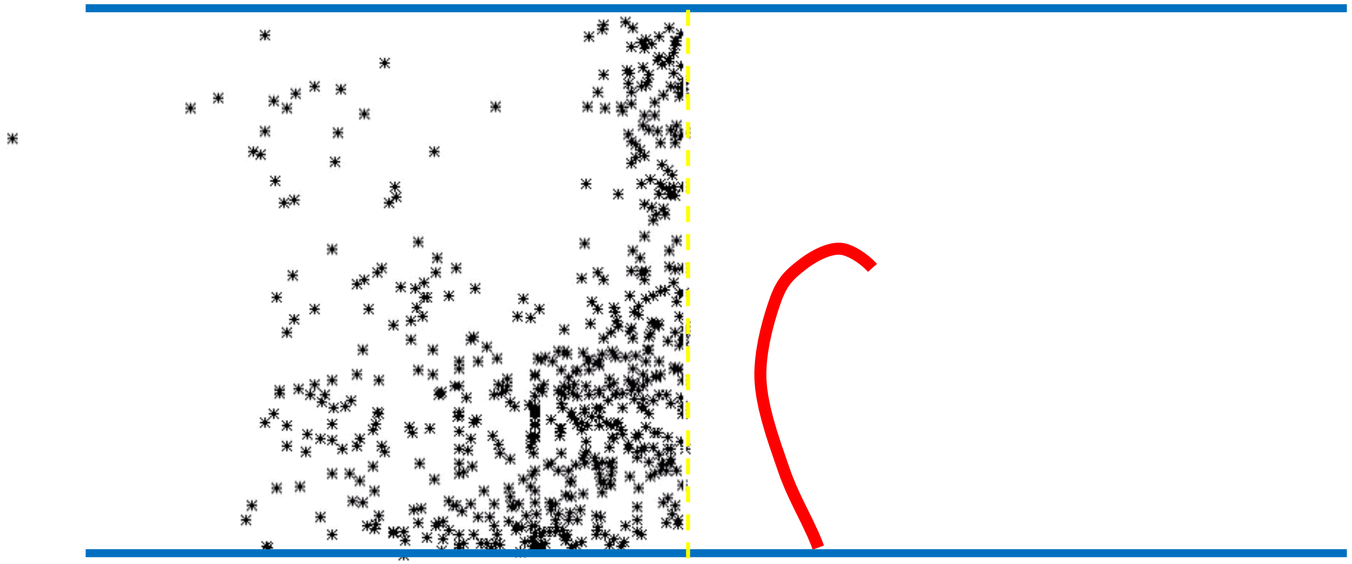
Invariant manifolds and barriers blocking swimming microbes in vortex flows

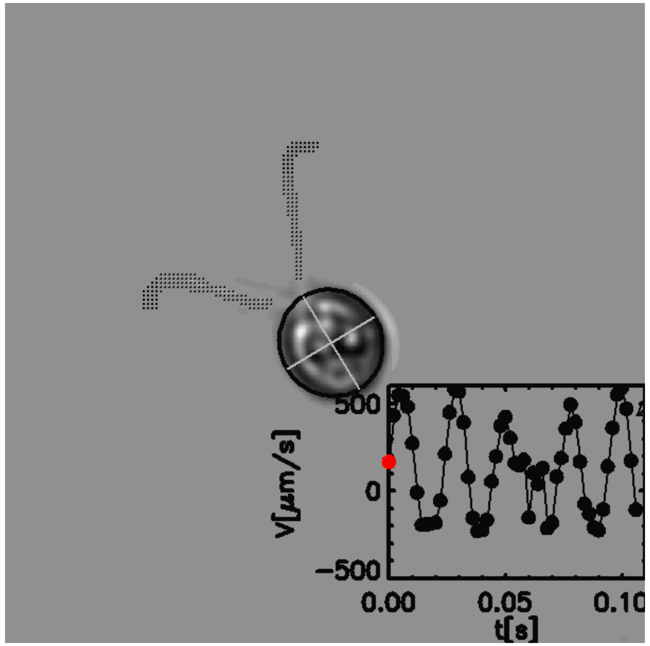
Cameron Lodi[*], Tom Solomon[*], Simon Berman[#] and Kevin Mitchell[#]
[*]Bucknell University and [#]UC-Merced



Key idea: Common theory of active mixing that applies to both propagating reaction fronts and self-propelled tracers in fluid flows.

Supported by NSF Grants DMR-1806355, CMMI-1825379 and S-STEM Program #1742124

How is a swimming microbe like a forest fire?



Chlamydomonas reinhardtii (green algae)
(From Gollub research group)

$\stackrel{?}{=}$

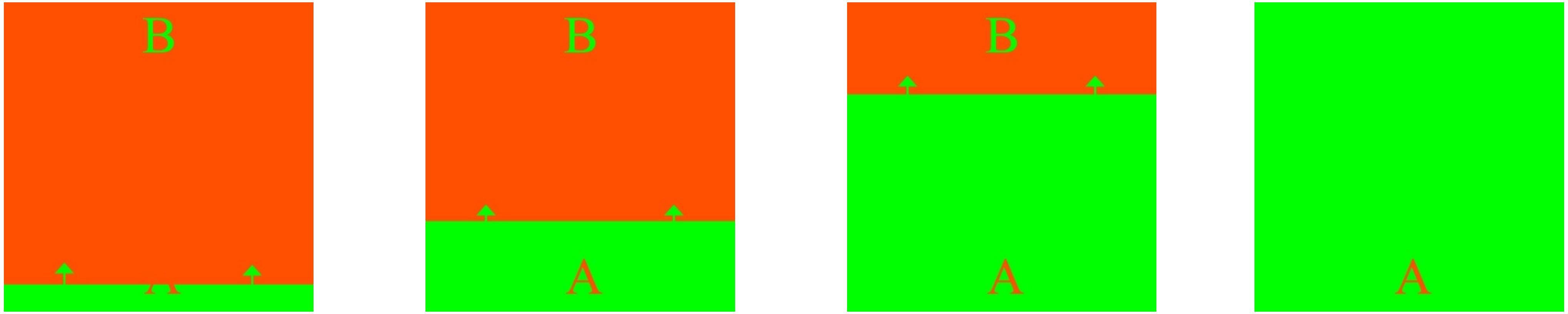
Forest fire in Brazil, image from BBC news, 9/2/1999

Front propagation

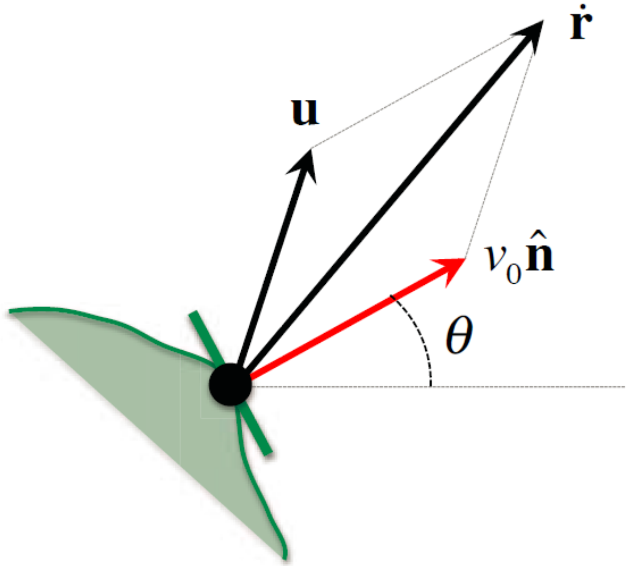
Examples

- Combustion reactions
- Bio systems: plankton blooms, spreading diseases, predator-prey systems, embryonic processes?
- Solidification

Forest fire in Brazil, image from BBC news, 9/2/1999



Theory for reaction fronts in fluid flows → Advection Reaction Diffusion

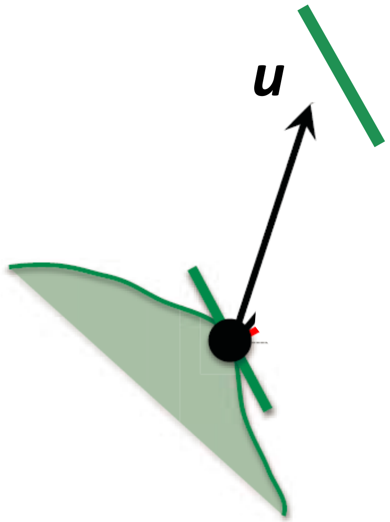


Approach: model the evolution of the front
→ 3 variables (x,y, θ) that describe a piece of the front

$$\dot{x} = u_x + v_0 cos\theta$$
$$\dot{y} = u_y + v_0 sin\theta$$
$$\dot{\theta} = 2u_{x,x} cos\theta sin\theta - u_{x,y} cos^2\theta + u_{y,x} sin^2\theta$$

Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

Advection Reaction Diffusion



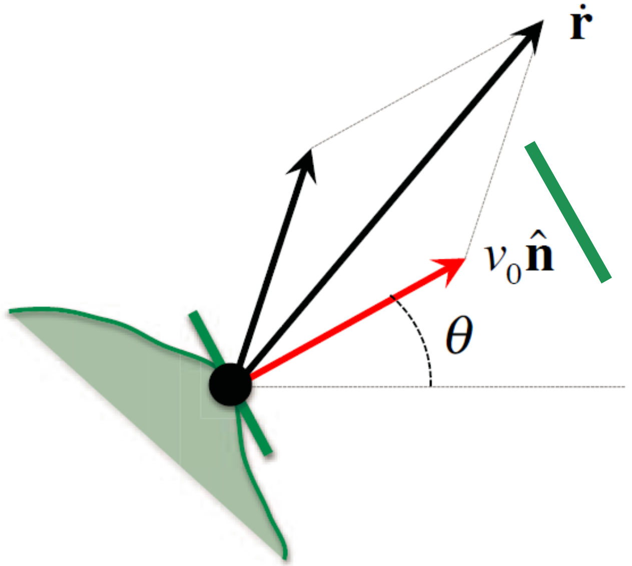
$$\dot{x} = u_x(x, y, t)$$
$$\dot{y} = u_y(x, y, t)$$

Front element can
→ be advected by the flow

u: flow velocity
v_o: ratio between reaction speed without a flow and flow velocity
θ : local angle of front

Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

Advection Reaction Diffusion



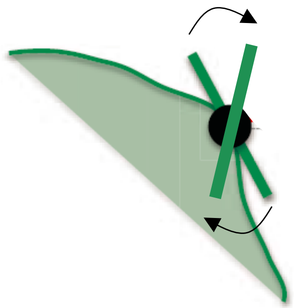
$$\dot{x} = \qquad + v_0 \cos\theta$$
$$\dot{y} = \qquad + v_0 \sin\theta$$

Front element can
→ Propagate (burn) relative to the flow in $\hat{n}$ direction

u: flow velocity
V_o: ratio between reaction speed without a flow and flow velocity
θ : local angle of front

Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

Advection Reaction Diffusion



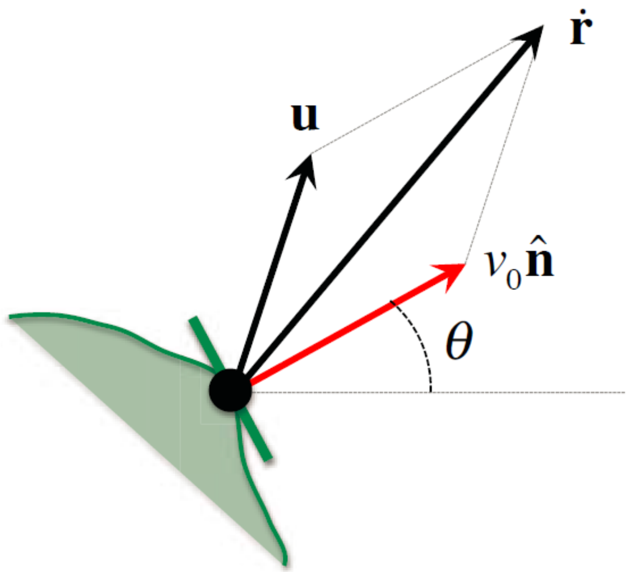
Front element can be
→ be rotated by the flow

u: flow velocity
v_o: ratio between reaction speed without a flow and flow velocity
θ : local angle of front

$$\dot{\theta} = 2u_{x,x}\sin\theta\cos\theta - u_{x,y}\cos^2\theta + u_{y,x}\sin^2\theta$$

Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

Advection Reaction Diffusion



Result is a 3D phase space through which front evolves

u: flow velocity

v_o: ratio between reaction speed without a flow and flow velocity

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$$\dot{x} = u_x + v_0 cos\theta$$
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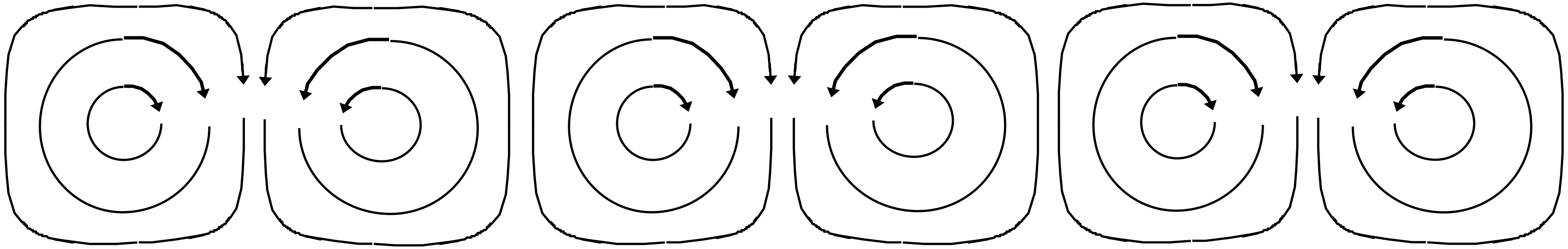
Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

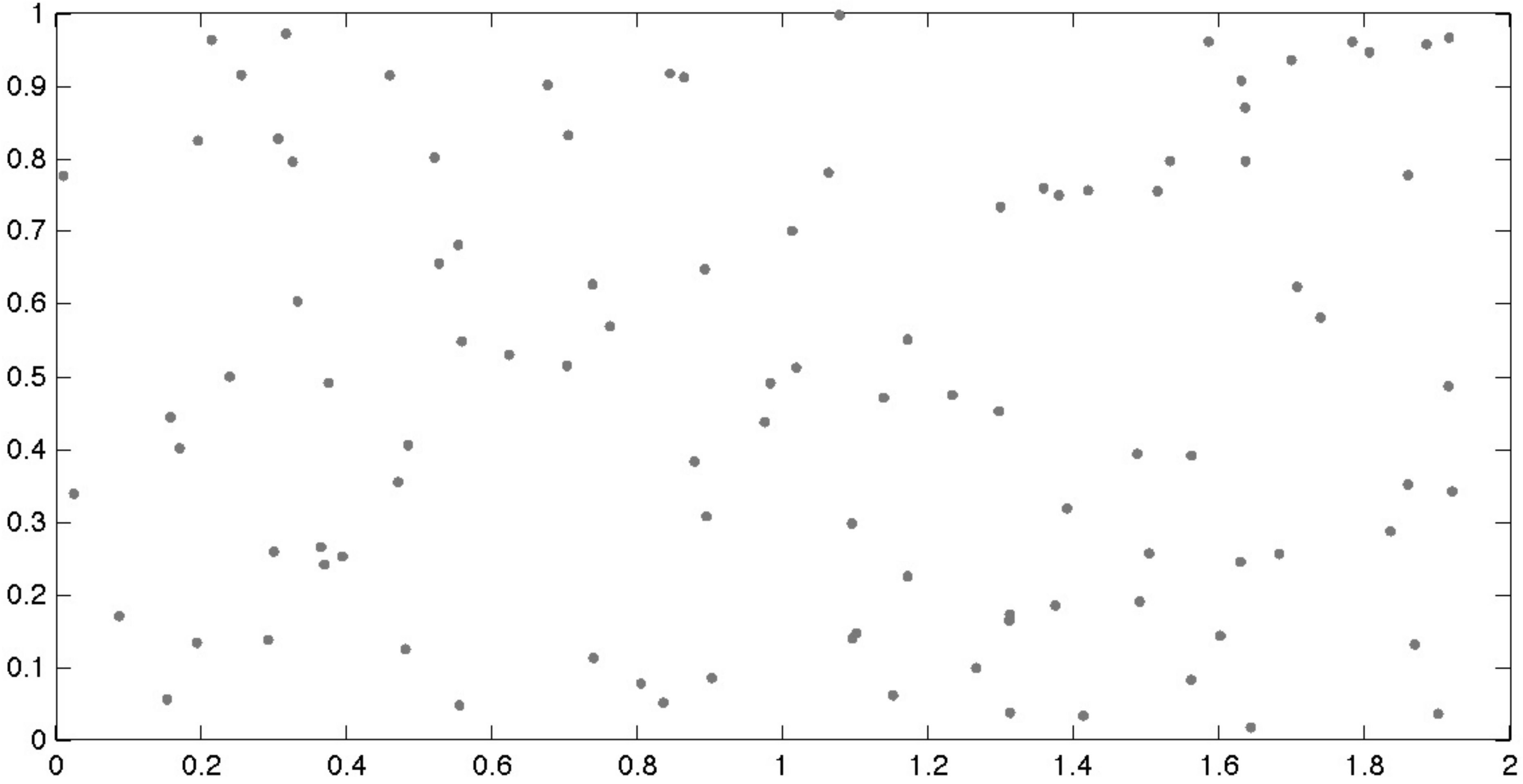
Counter-rotating vortex chain flow



$$u_x = \frac{k_y}{k_x} U cos(k_x x) \sin(k_y y)$$
$$u_y = -U sin(k_x x) \cos(k_y y)$$

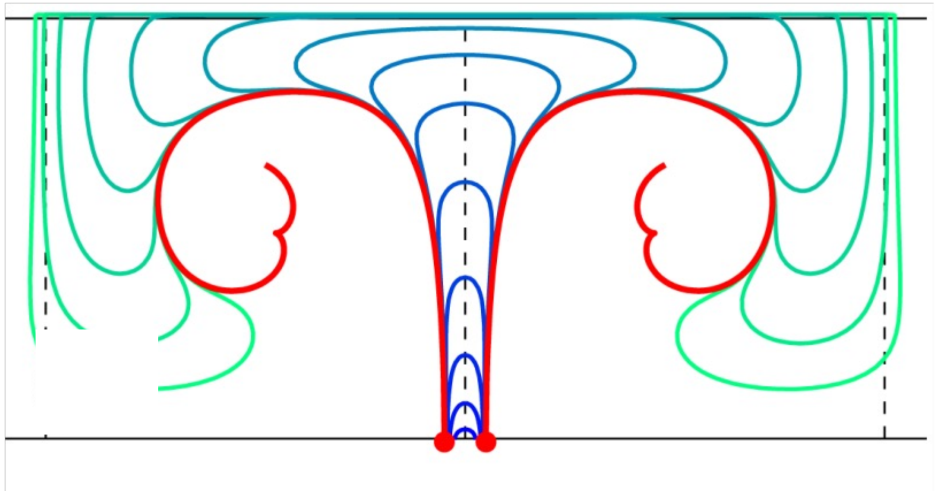
(for free-slip boundary conditions)

Burning invariant manifolds (BIMs) as one-way barriers to fronts

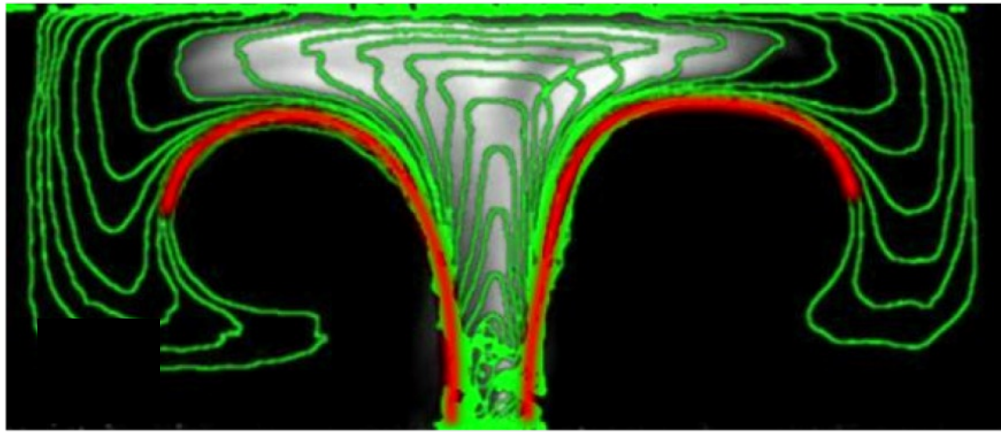


Simulations by John Mahoney & Kevin Mitchell

Reaction triggered at vortex corner blocked by BIMs on both sides



Theory

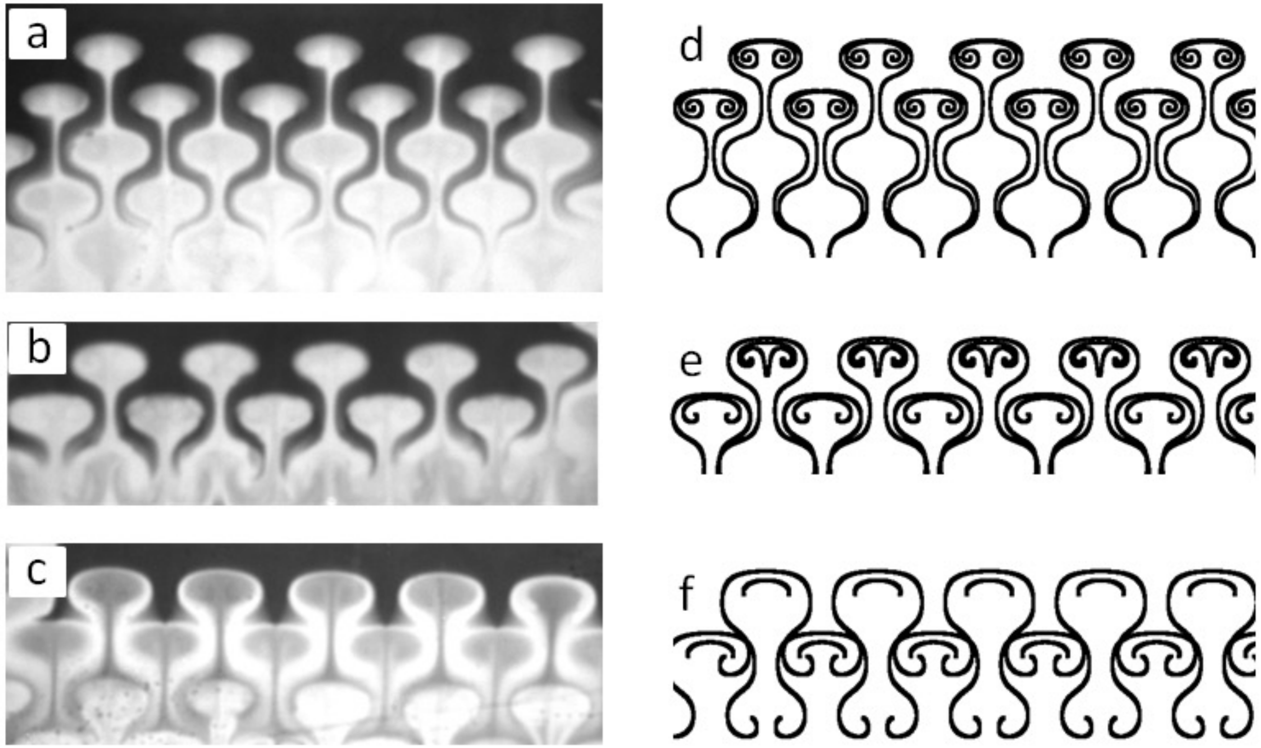


Experiments

Mahoney, Bargteil, Kingsbury, Mitchell, and Solomon. EPL (2012)

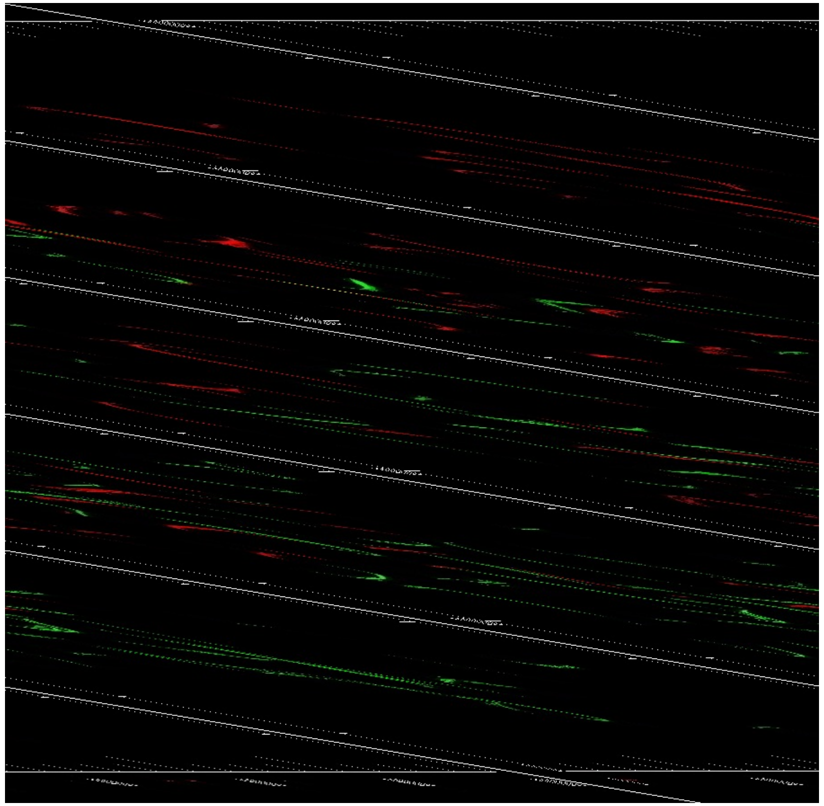
Burning invariant manifolds (BIMs) measured experimentally in a wide range of laminar flows

Pinning of reaction fronts due to BIMs in vortex flows with imposed winds.



Megson, Lilienthal & Solomon, Phys. Fluids (2015)

BIMs as reaction barriers in spatially-disordered flows.

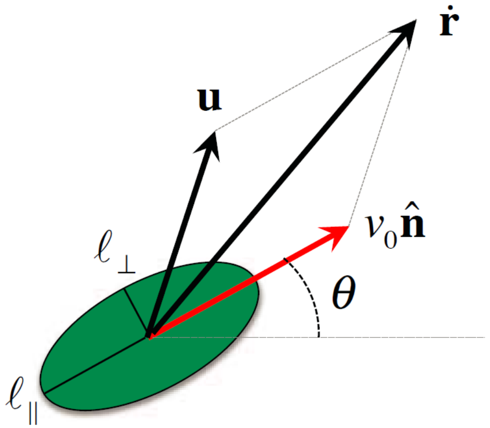


Mahoney, Bargteil, Kingsbury, Mitchell and Solomon, EPL (2012); Bargteil & Solomon, Chaos (2012).

How Do We Apply it to Swimming Microbes?

Theory for Swimmers

Same as for chemical reactions, but modified to account for shape of organism



u: flow velocity

v_o: ratio between swimming speed without a flow and characteristic flow speed

θ : local angle of swimmer

$$\alpha = \frac{\gamma^2 - 1}{\gamma^2 + 1}$$

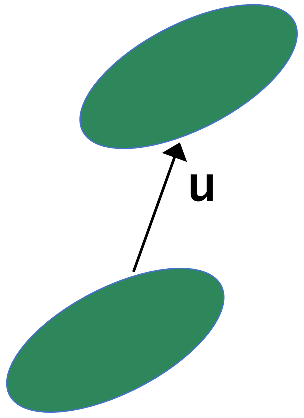
where $\gamma = l_{||}/l_{\perp}$ is the aspect ratio of the swimmer.

$$\dot{x} = u_x + v_0 cos\theta$$
$$\dot{y} = u_y + v_0 sin\theta$$
$$\dot{\theta} = (1 + \alpha)\left(\frac{\omega_z}{2}\right) - \alpha(2u_{x,x}cos\theta sin\theta - u_{x,y}cos^2\theta + u_{y,x}sin^2\theta)$$

Berman, Buggeln, Brantley, Mitchell, and Solomon, Phys. Rev. Fluids 6 L012501 (2021)

Theory for Swimmers

Swimmer can
→ be advected by the flow



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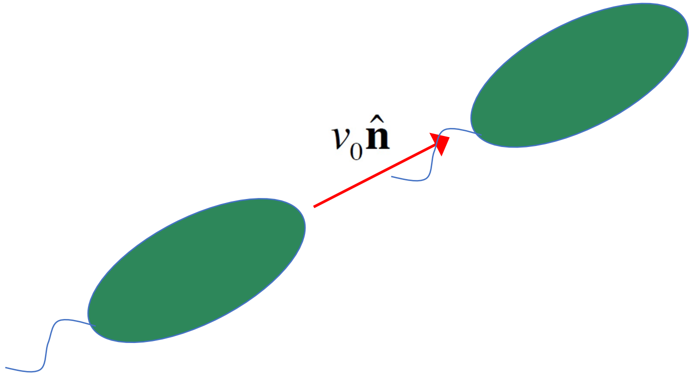
Berman, Buggeln, Brantley, Mitchell, and Solomon, Phys. Rev. Fluids 6 L012501 (2021)

Theory for Swimmers

Swimmer can

→ swim relative to the fluid



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Theory for Swimmers

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Swimmer can
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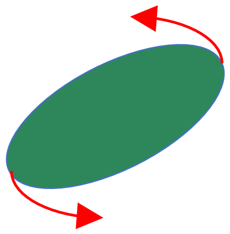

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Berman, Buggeln, Brantley, Mitchell, and Solomon, Phys. Rev. Fluids 6 L012501 (2021)

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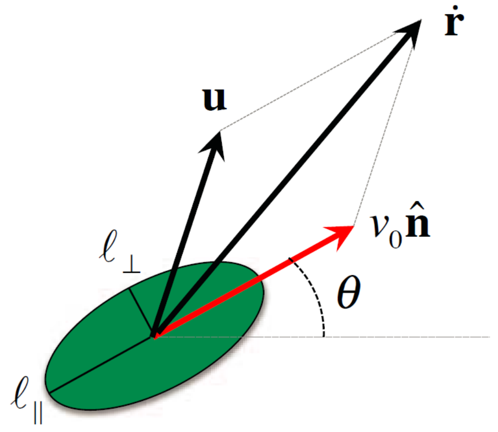

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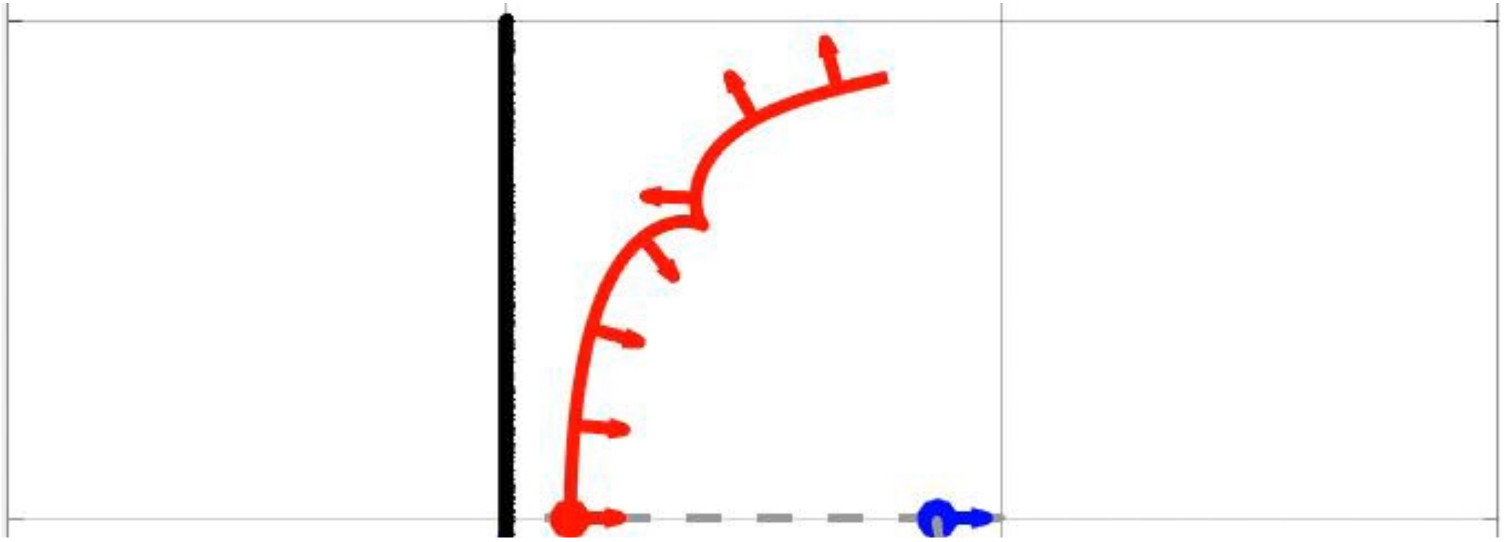
Berman, Buggeln, Brantley, Mitchell, and Solomon, Phys. Rev. Fluids 6 L012501 (2021)

Swimming invariant manifolds (SwIMs) as one-way barriers to fronts

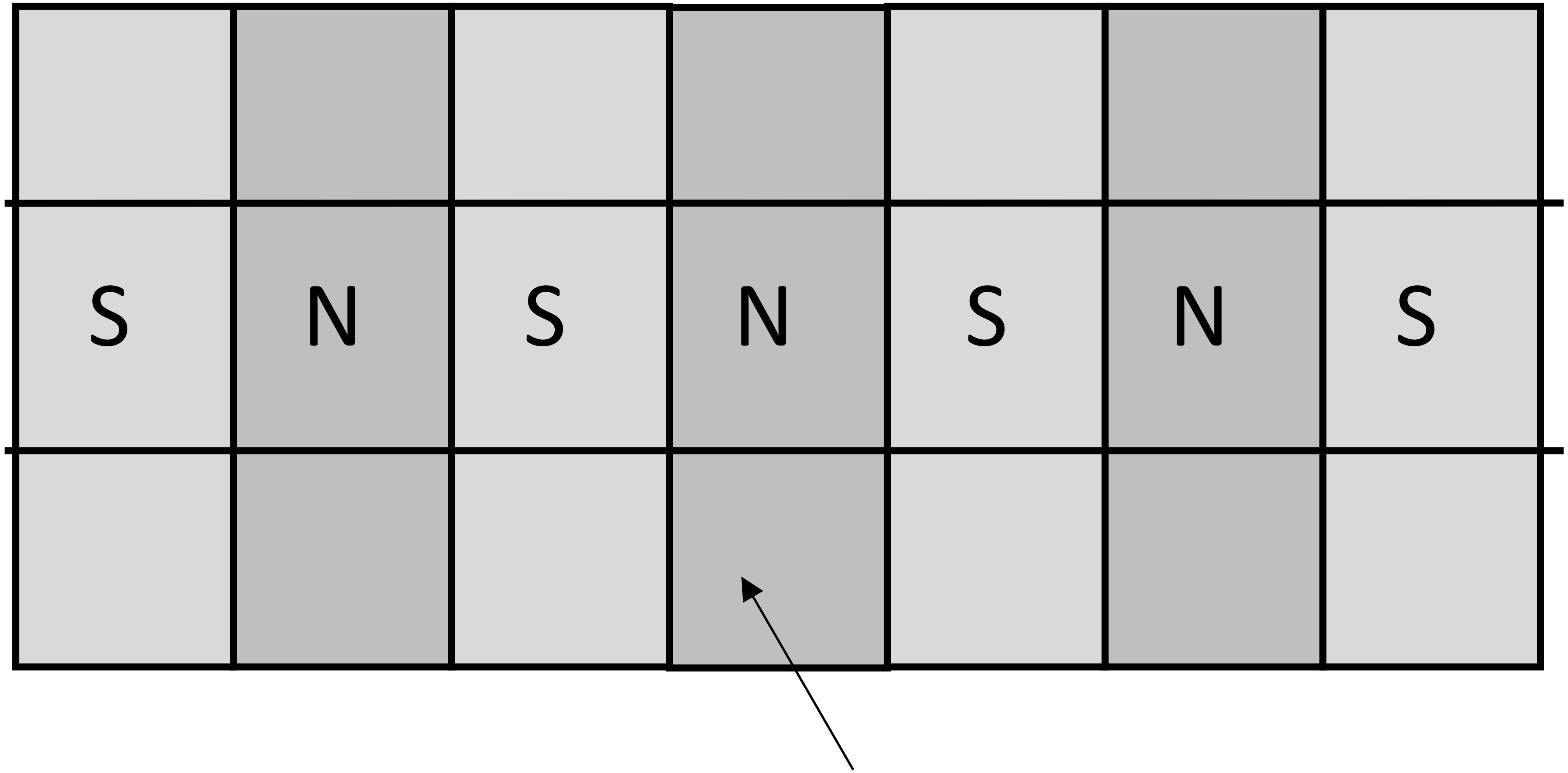


Simulation by Simon Berman

Magnetohydrodynamic Forcing

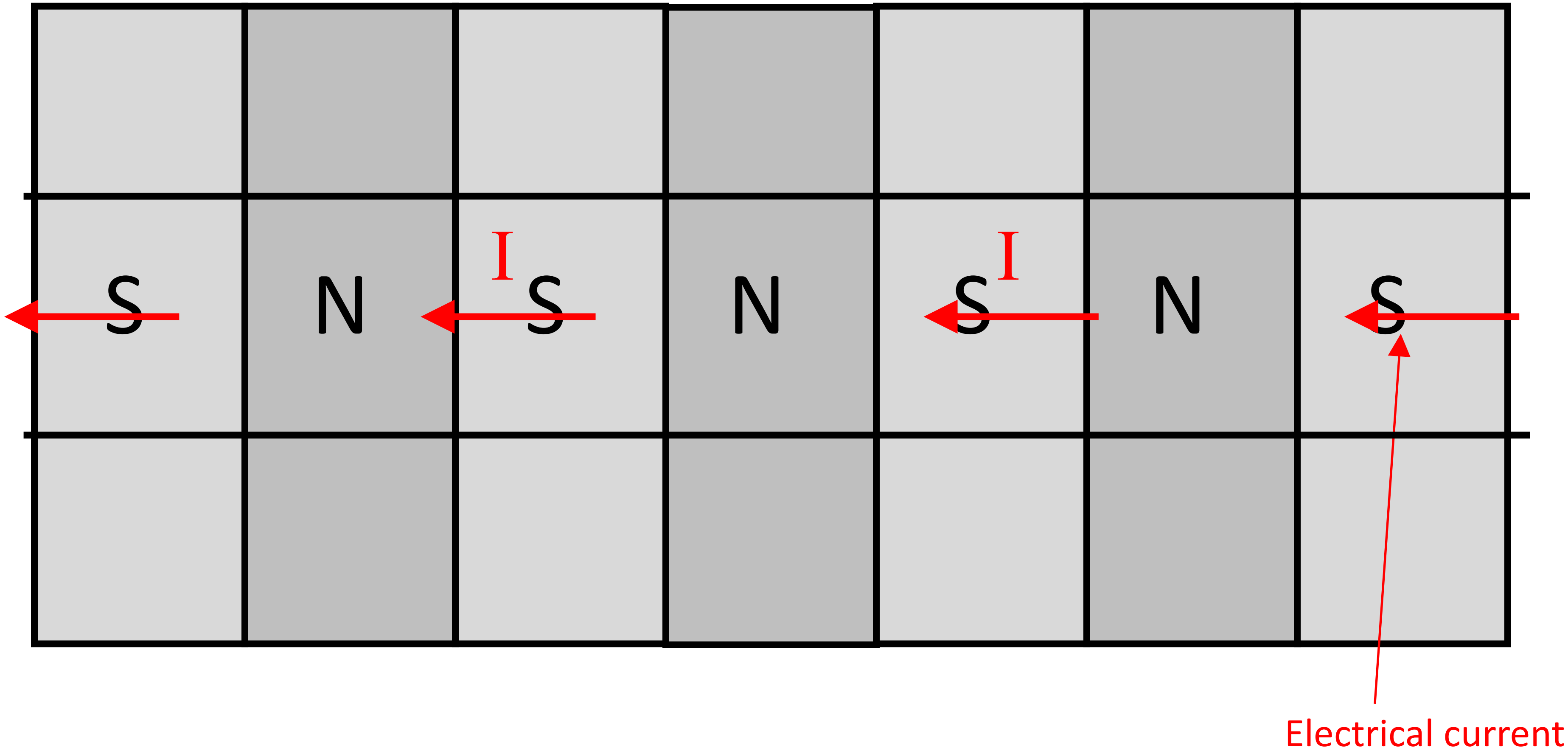
Acrylic channel with width 1.0 mm and depth 1.0 mm

Magnetohydrodynamic Forcing

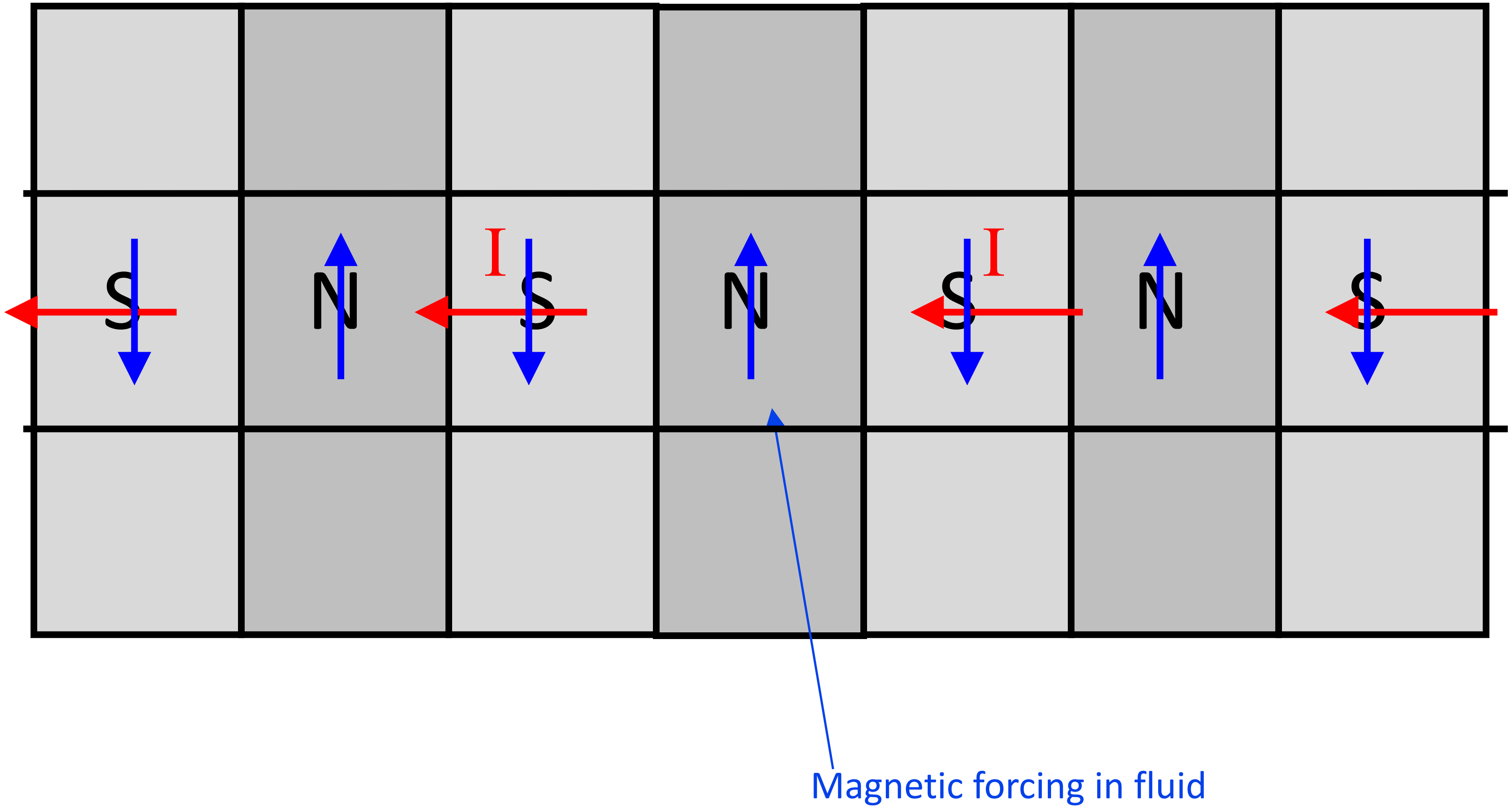


Strips of alternating magnets below the channel

Magnetohydrodynamic Forcing

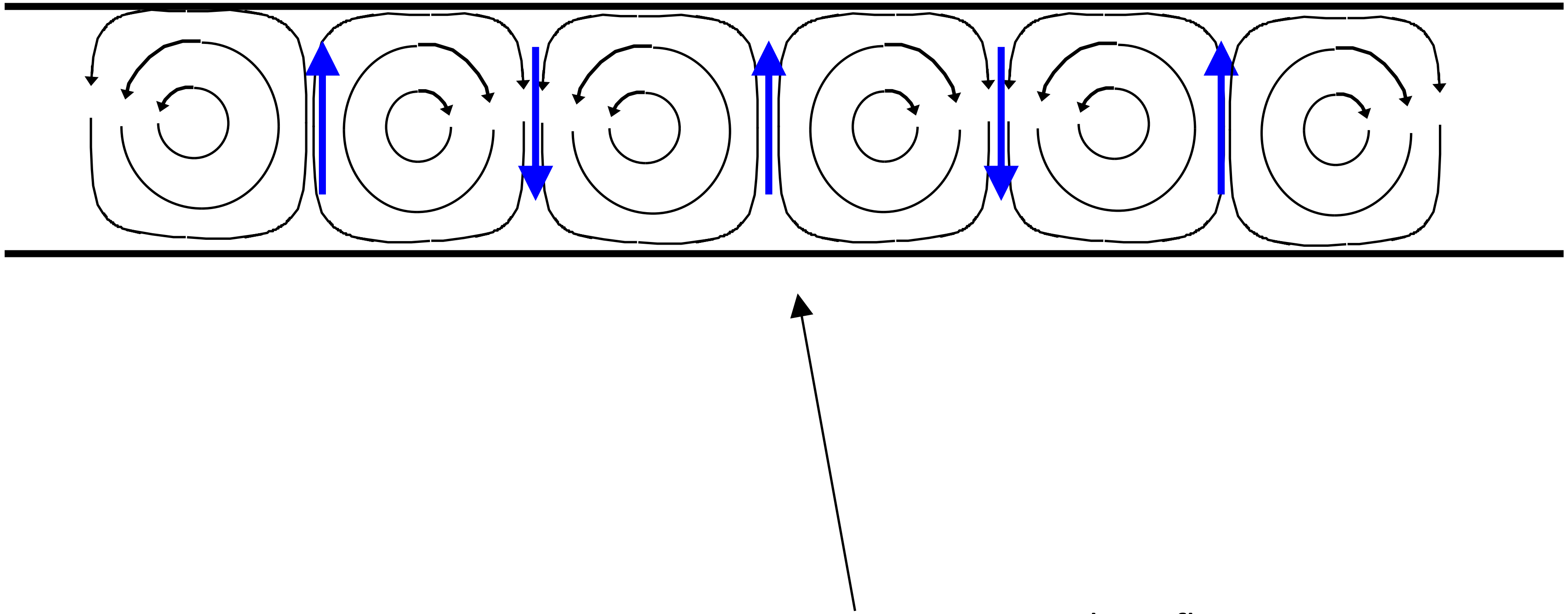


Magnetohydrodynamic Forcing



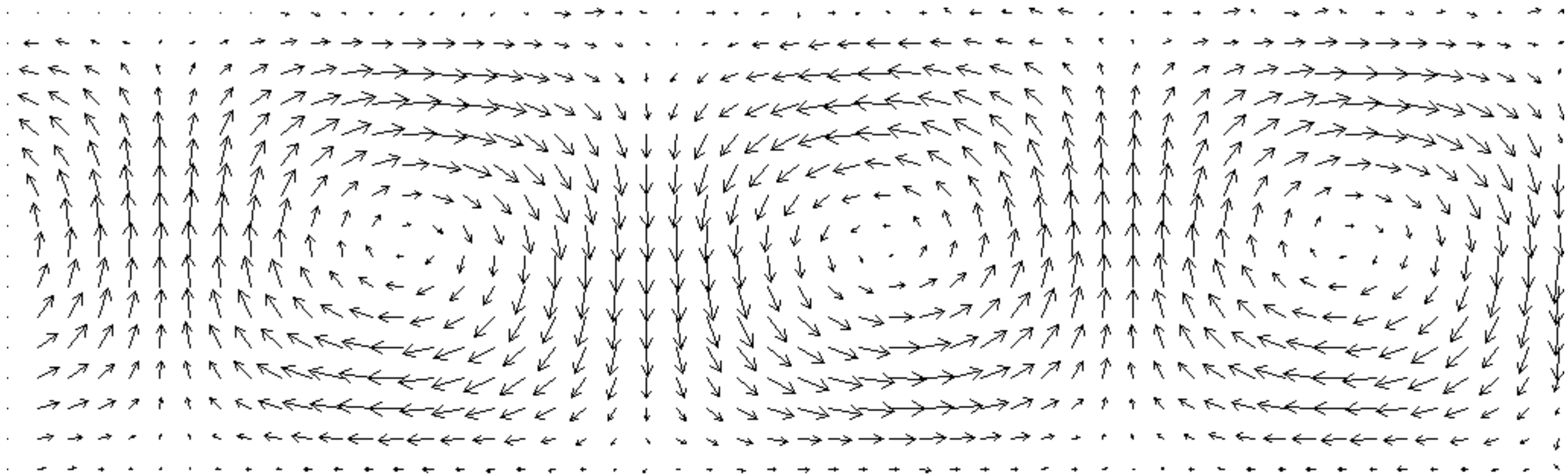
$$\vec{F} = q\ \vec{v} \times \vec{B}$$

Magnetohydrodynamic Forcing

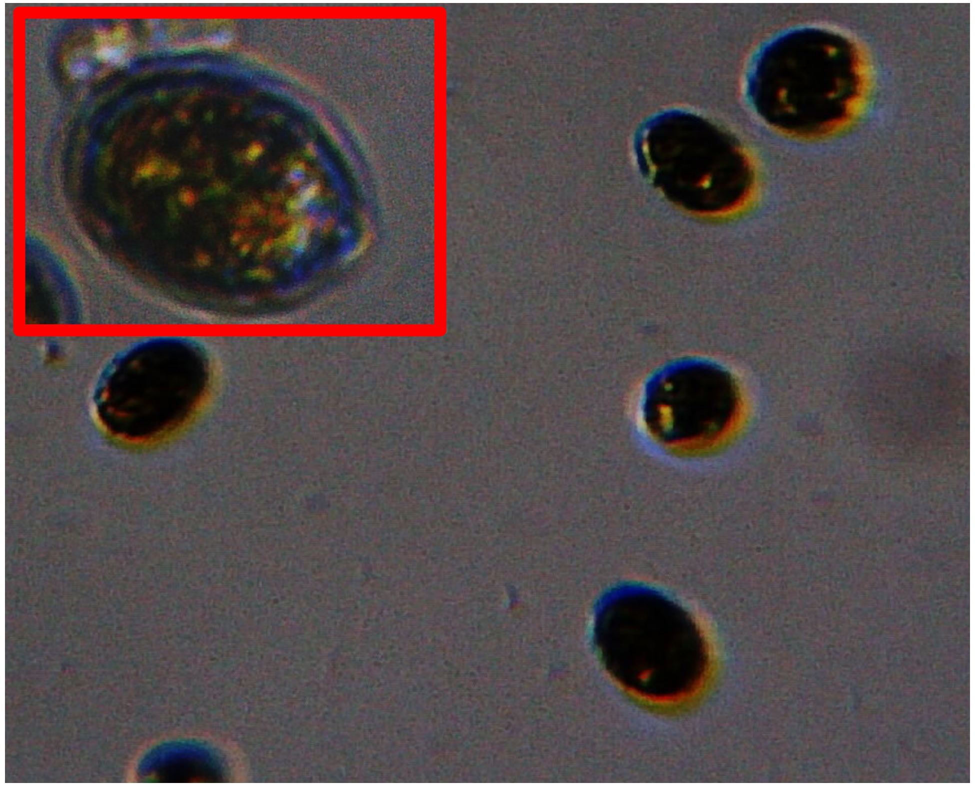


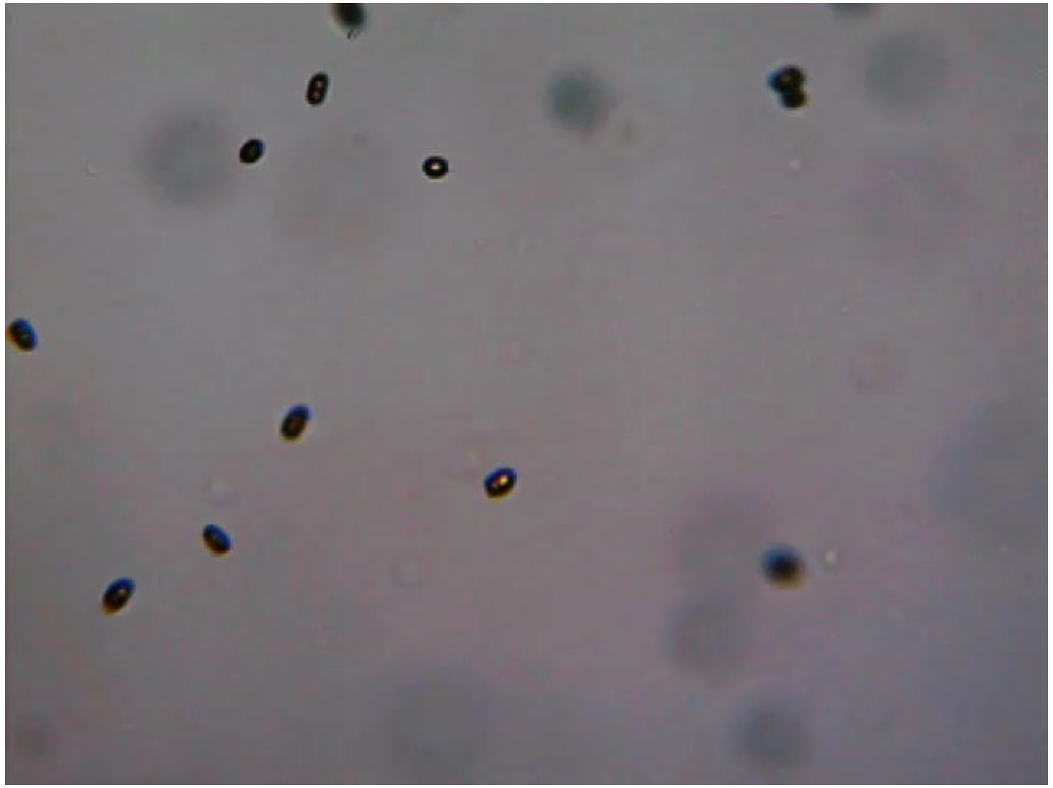
Vortex chain flow

Resulting velocity field



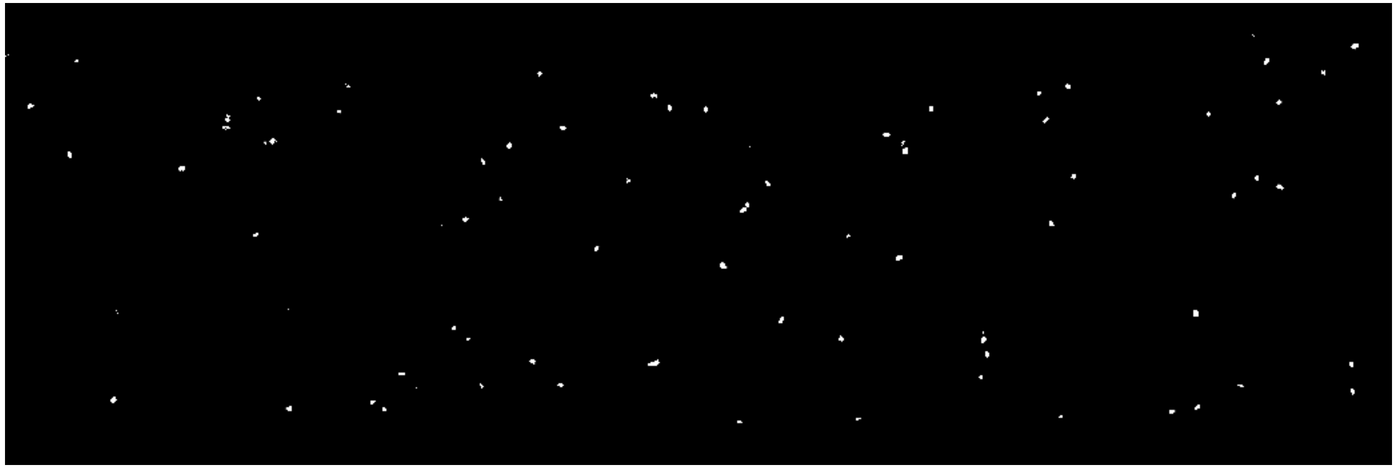
Swimming microbes: tetraselmis



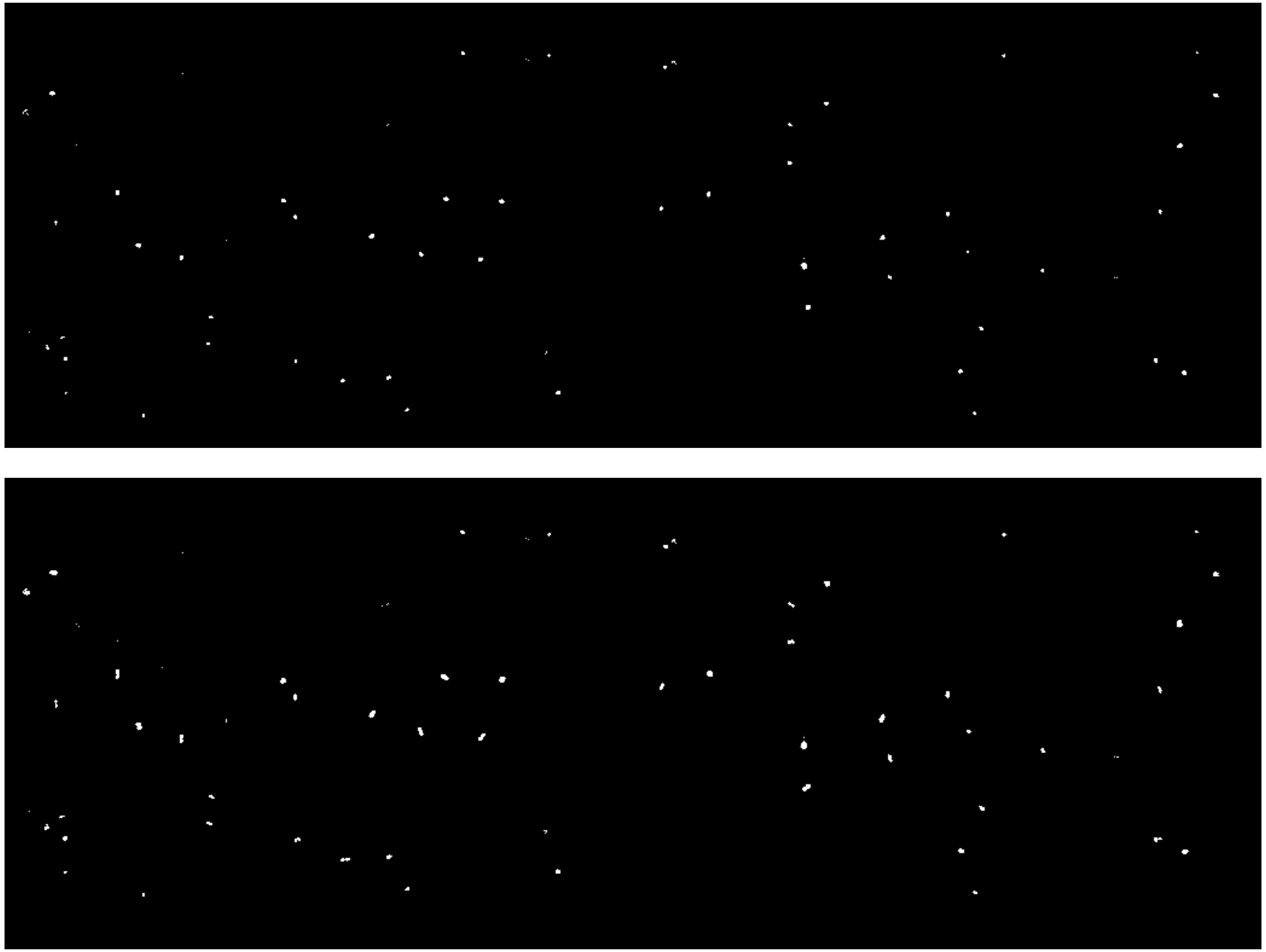


$\alpha \approx 0.3 - 0.4$
(typical swimming speeds
~ 100 – 150 μm/s)

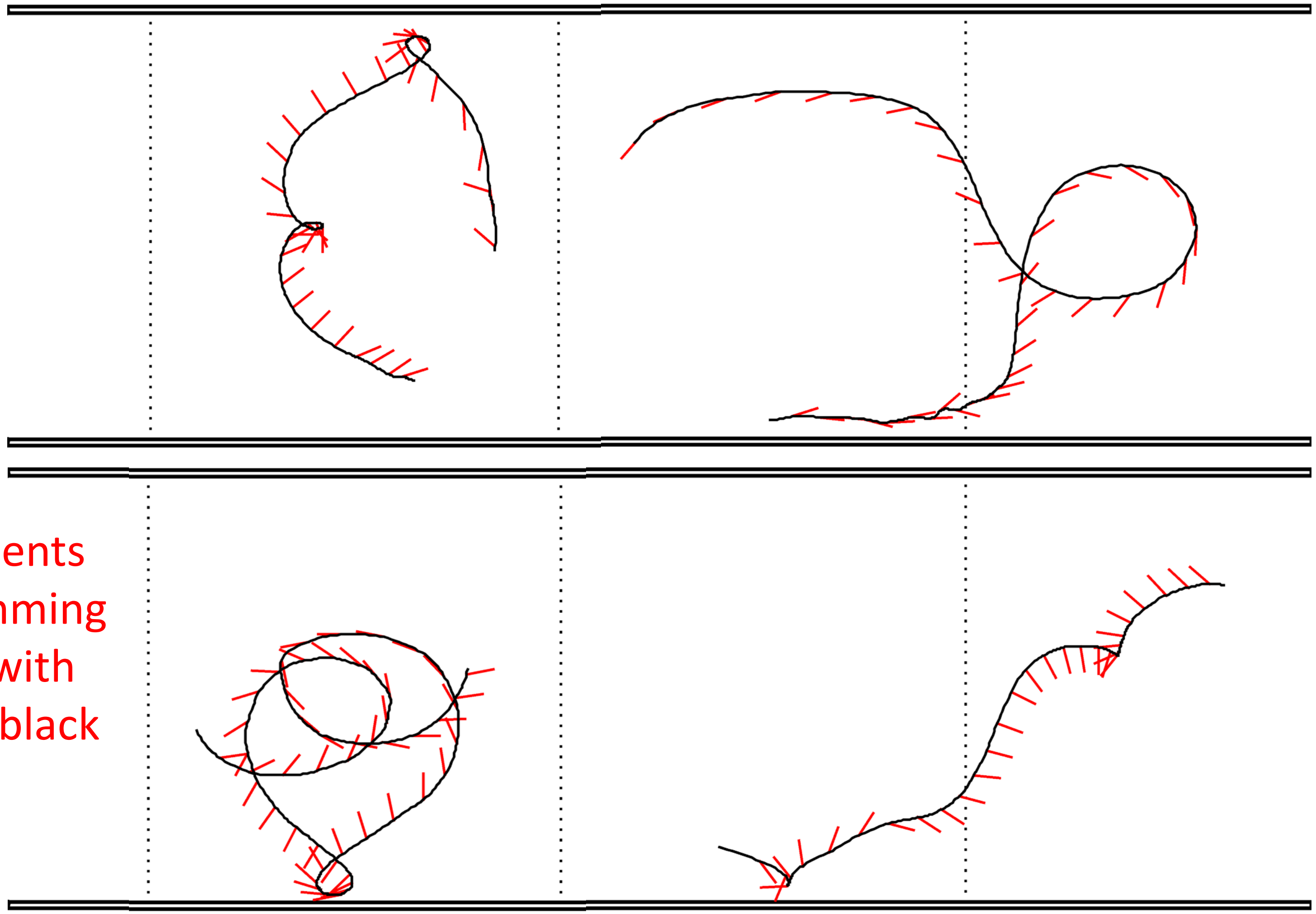
Tetraselmis swimming in channel, no imposed flow



Tetraselmis swimming in vortex flow, U = 370 μm/s

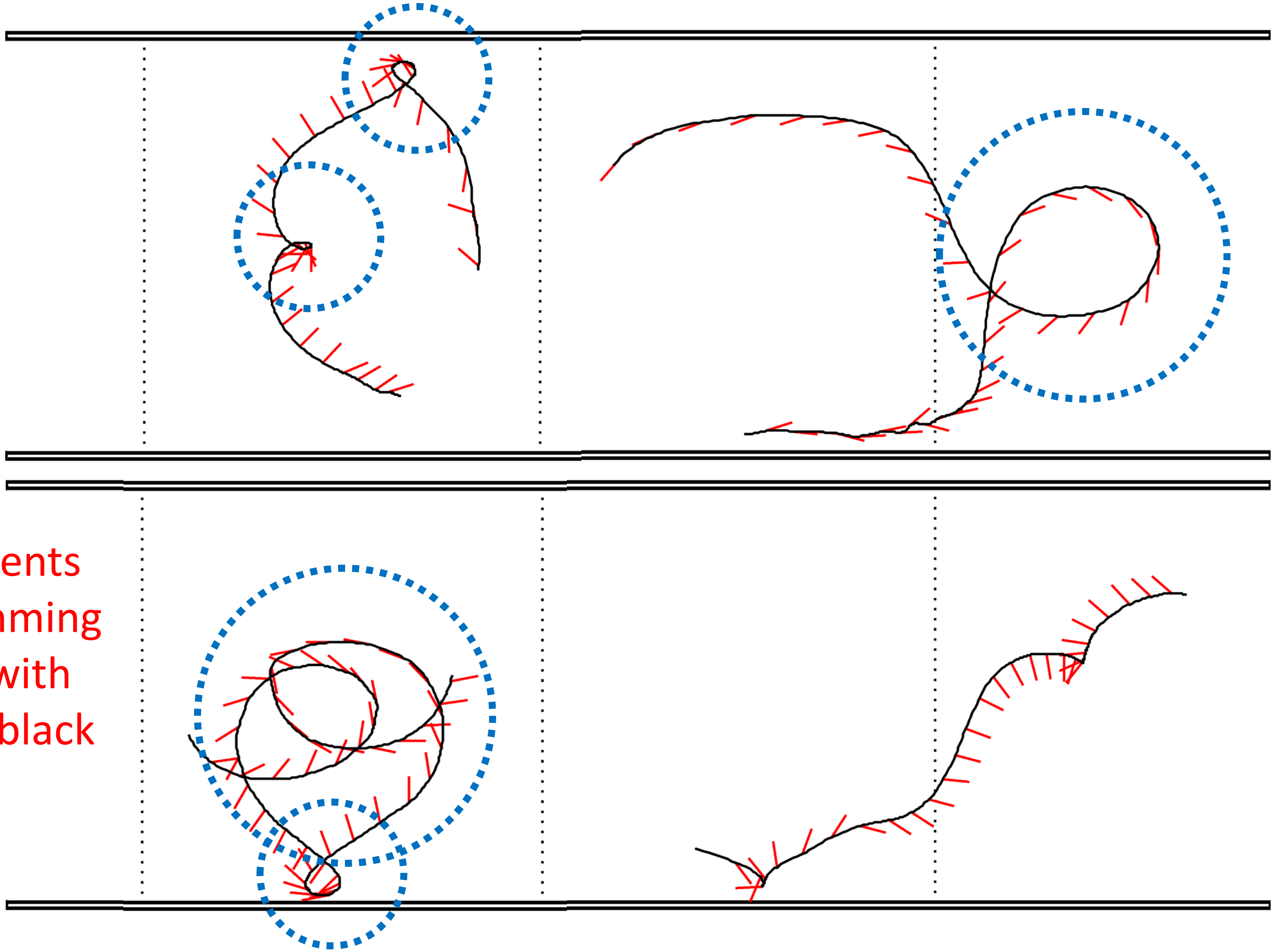


$v_0 = V_s/U = 0.5$



The red segments show the swimming orientation, with the tail at the black curve

$v_0 = V_s/U = 0.5$

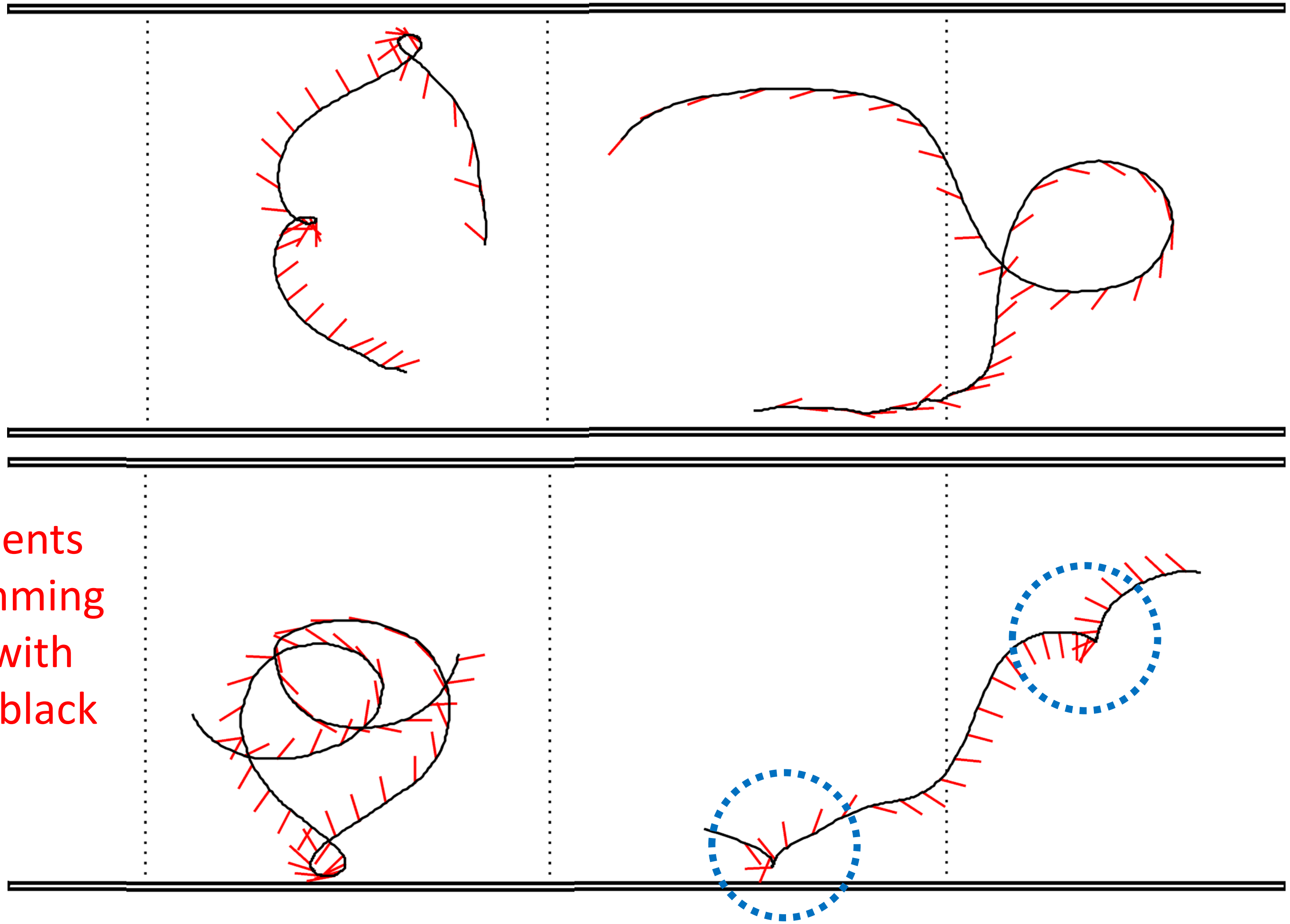


Common features

Loops, typically accompanied by a flip of 180° in swimming orientation.

The red segments show the swimming orientation, with the tail at the black curve

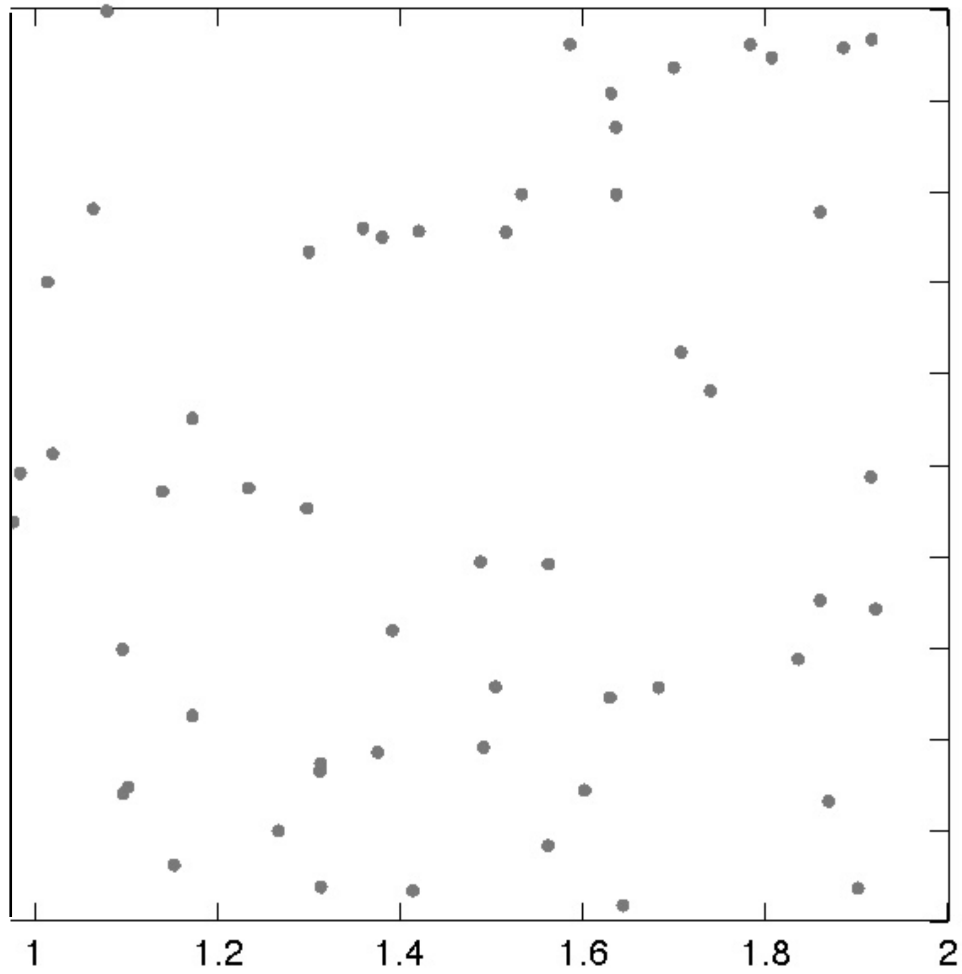
$v_0 = V_s/U = 0.5$



The red segments show the swimming orientation, with the tail at the black curve

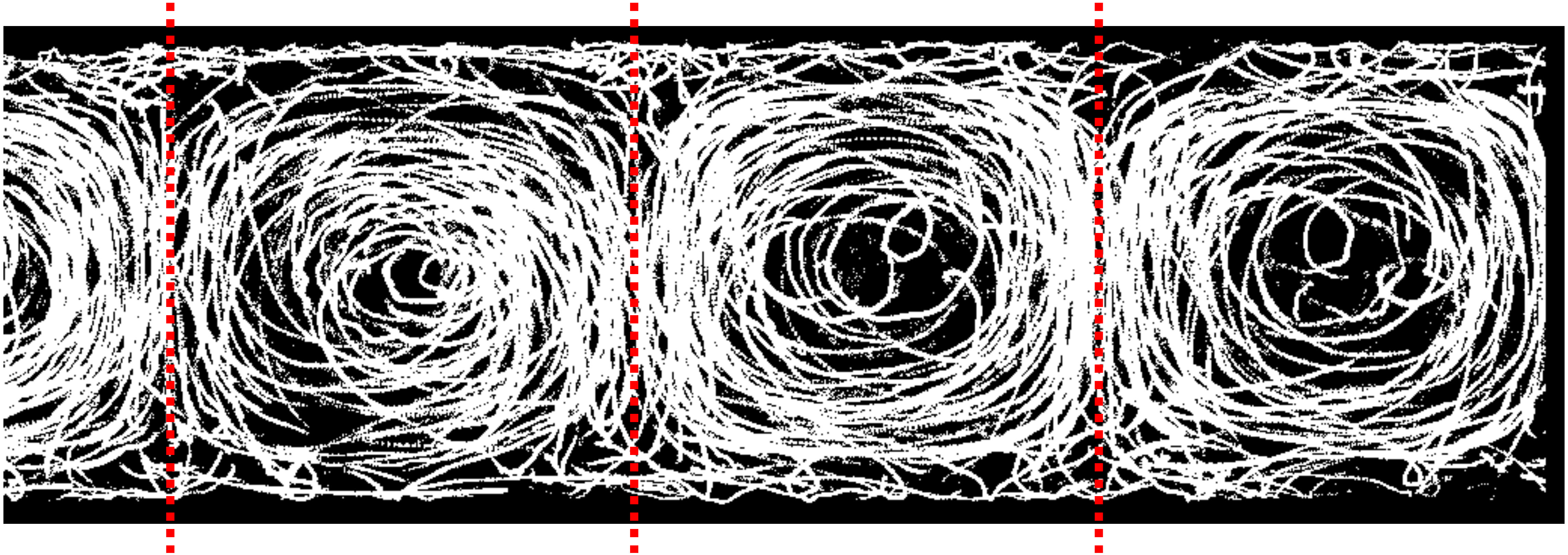
Common features

Cusps, also accompanied by a flip in orientation by 180°.

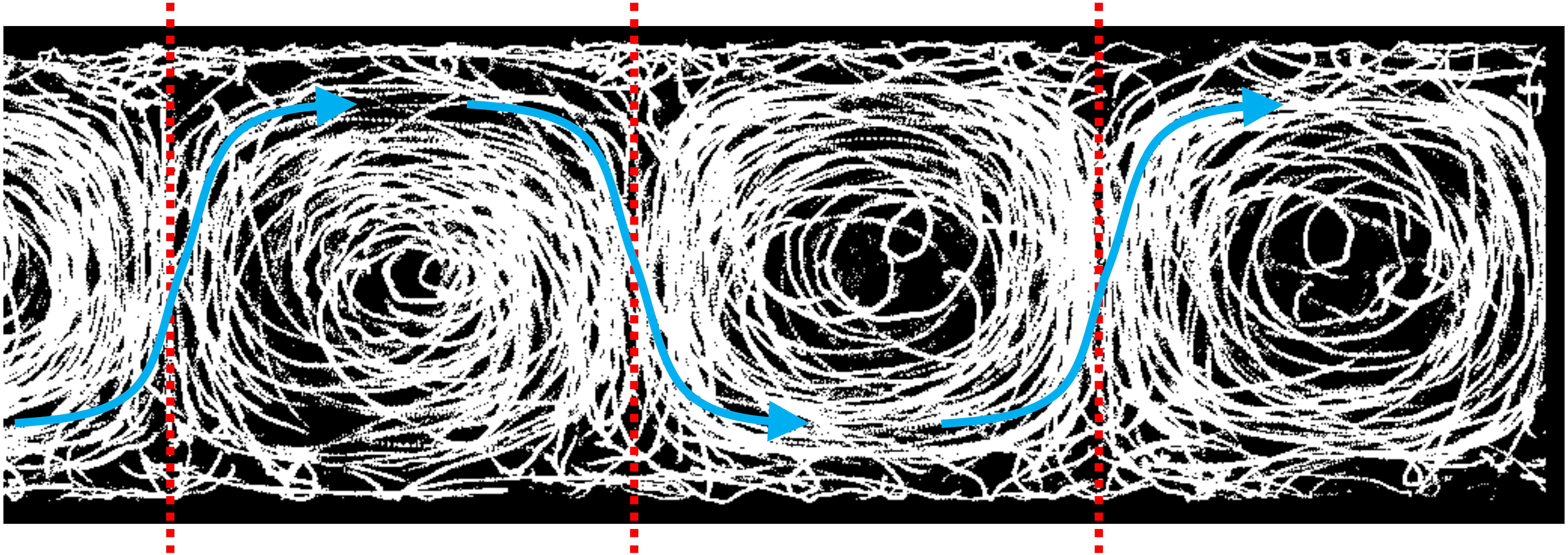


Cusps are a common feature between reaction fronts and swimming microbes – associated with flip in direction.

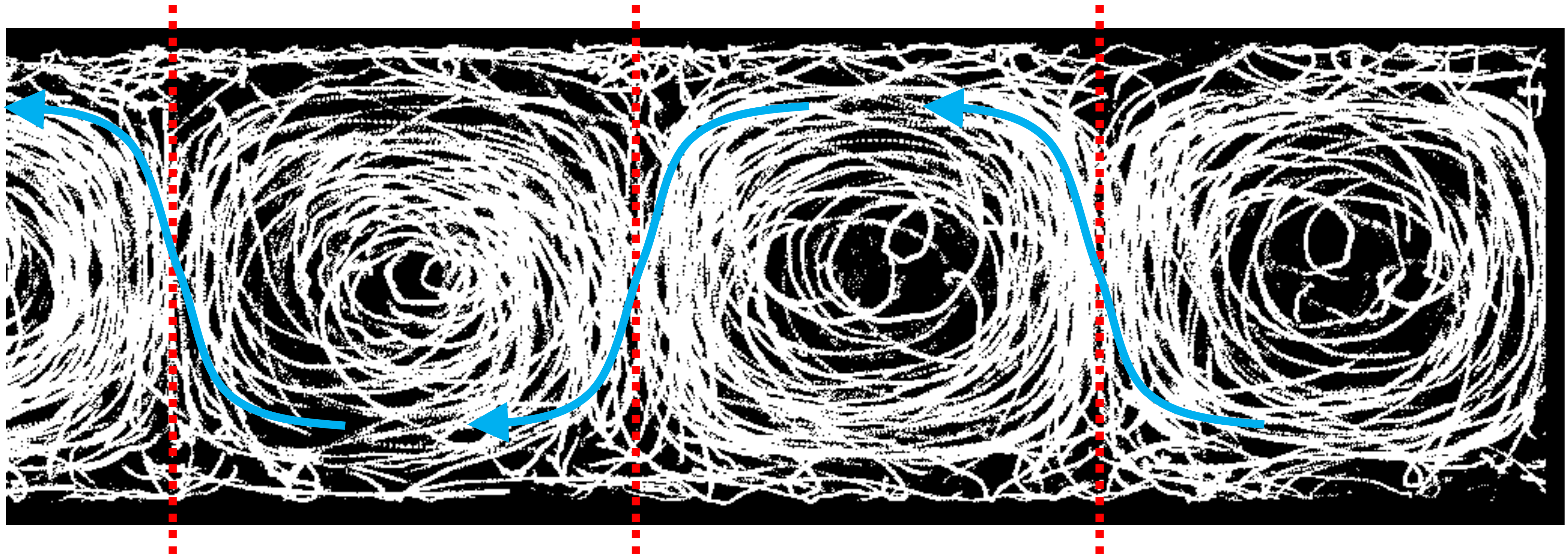
Simulations by John Mahoney & Kevin Mitchell



Can combine boundary-crossing trajectories from all three separatrices …

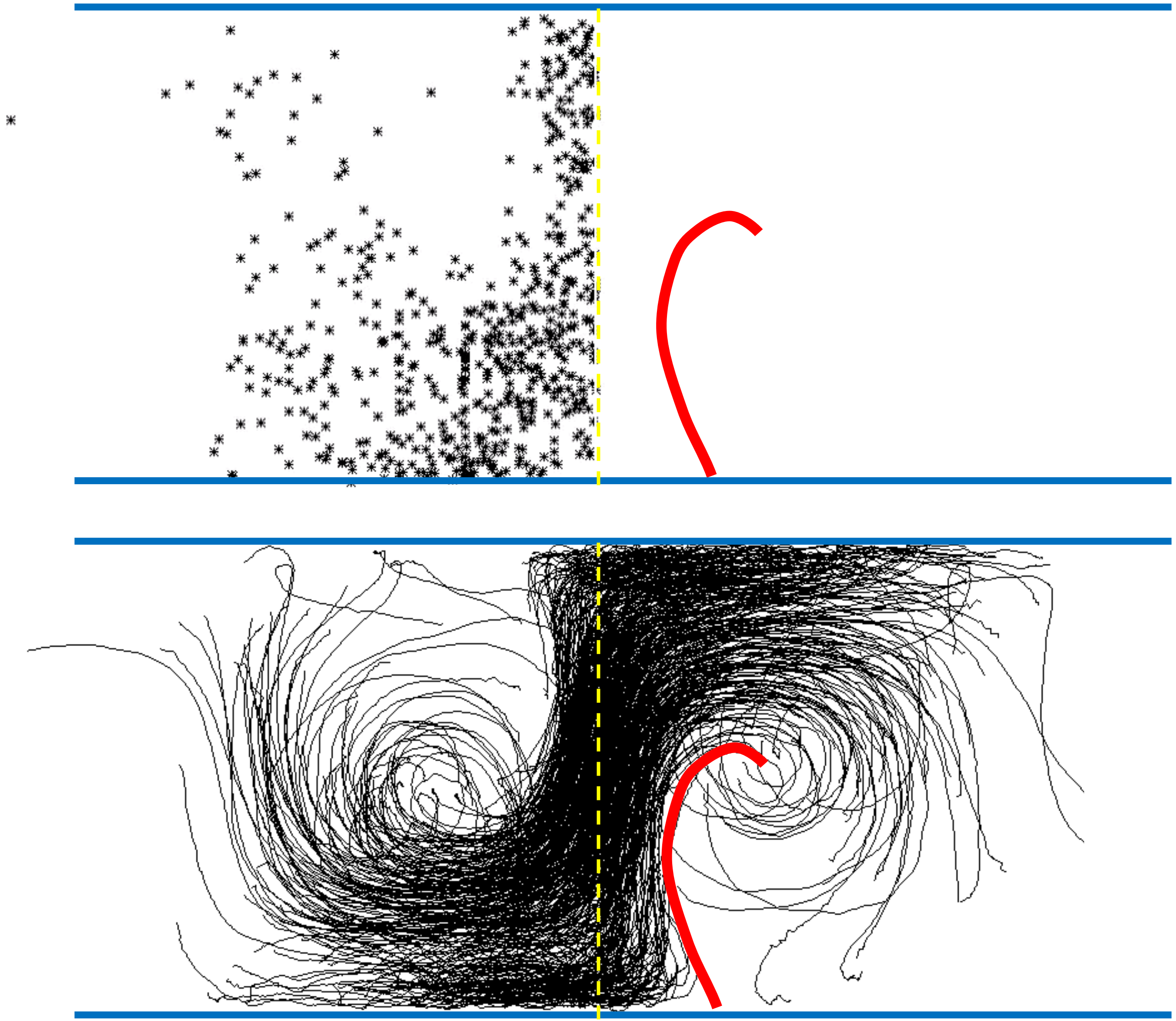


Tracers can cross vortex boundary going to the right …



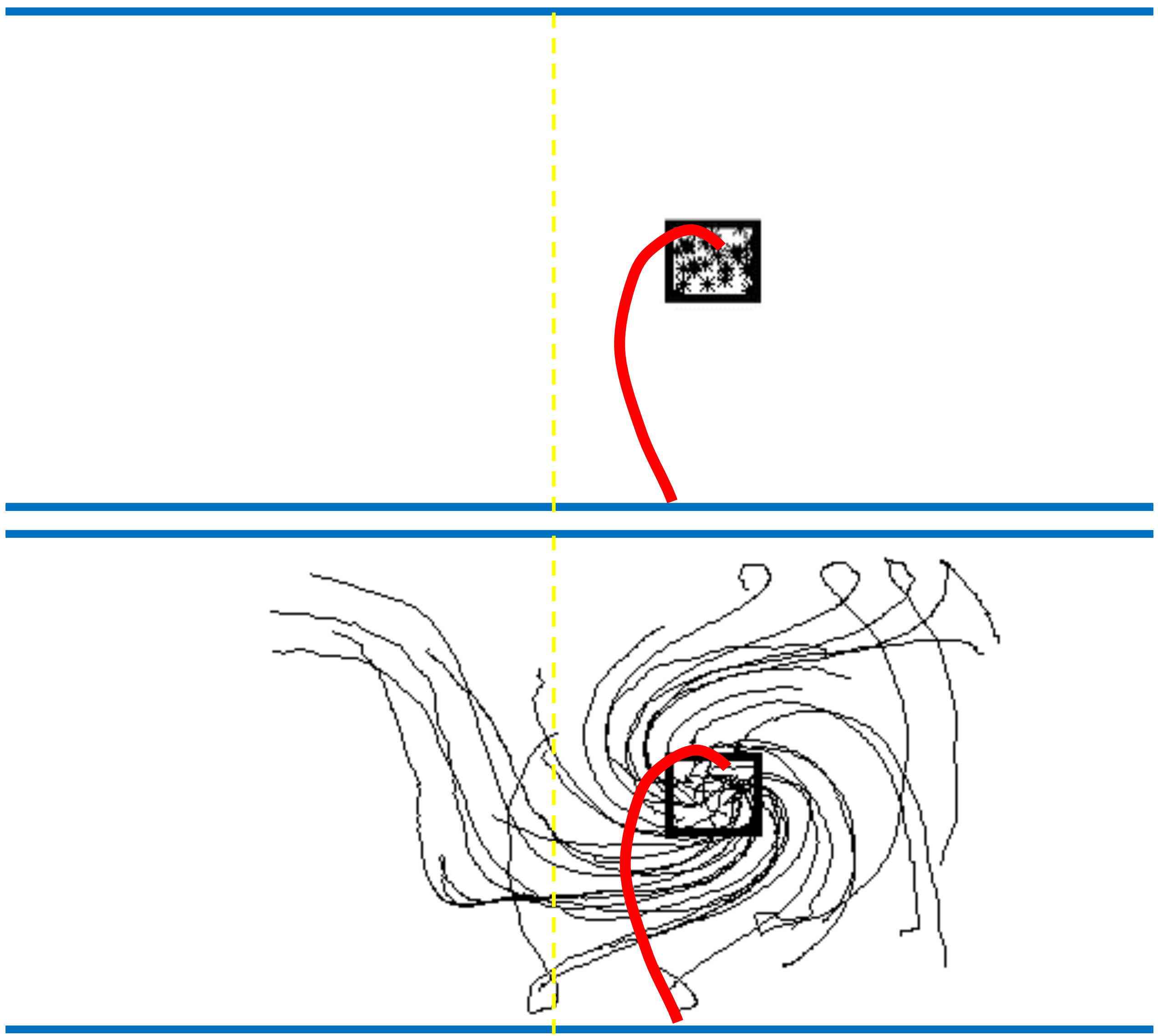
… or tracers can cross vortex boundary going to the left

$v_0 = V_s/U = 0.5$



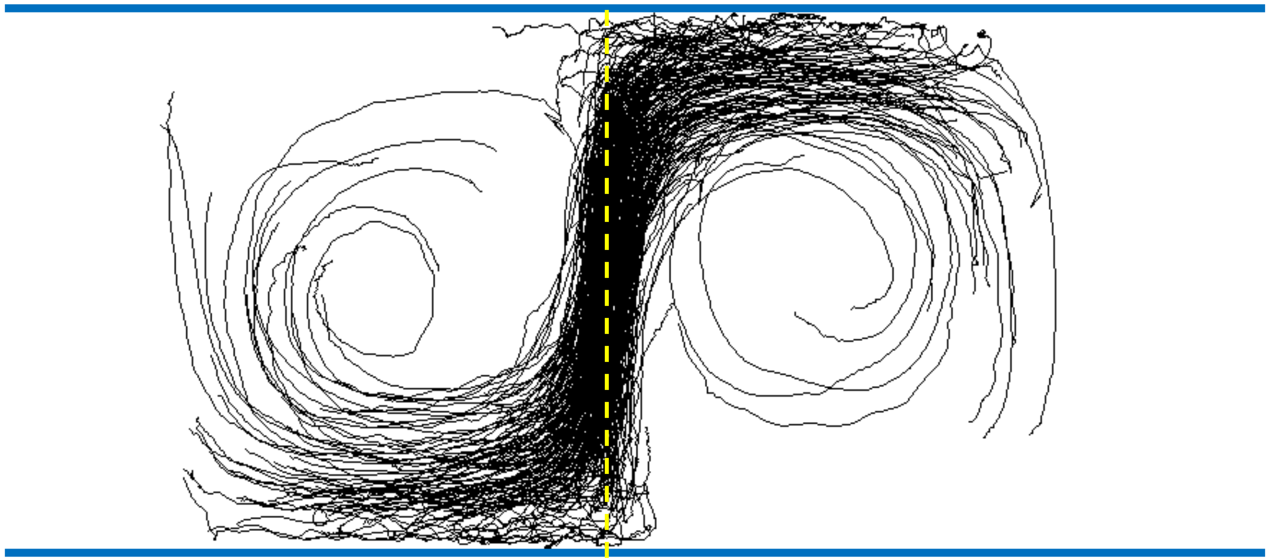
Experimentally, there are curves (SwIMs?) that block swimmers going to the right …

$v_0 = V_s/U = 0.5$

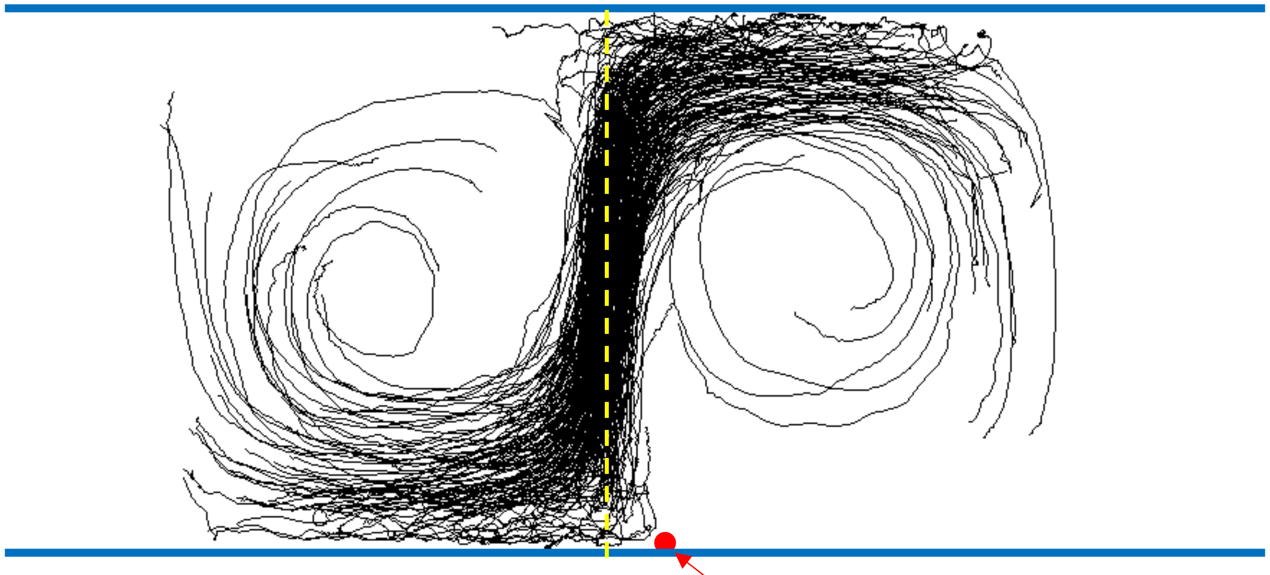


... but allow swimmers to cross freely in the other direction.

$v_0 = V_s/U = 0.3$

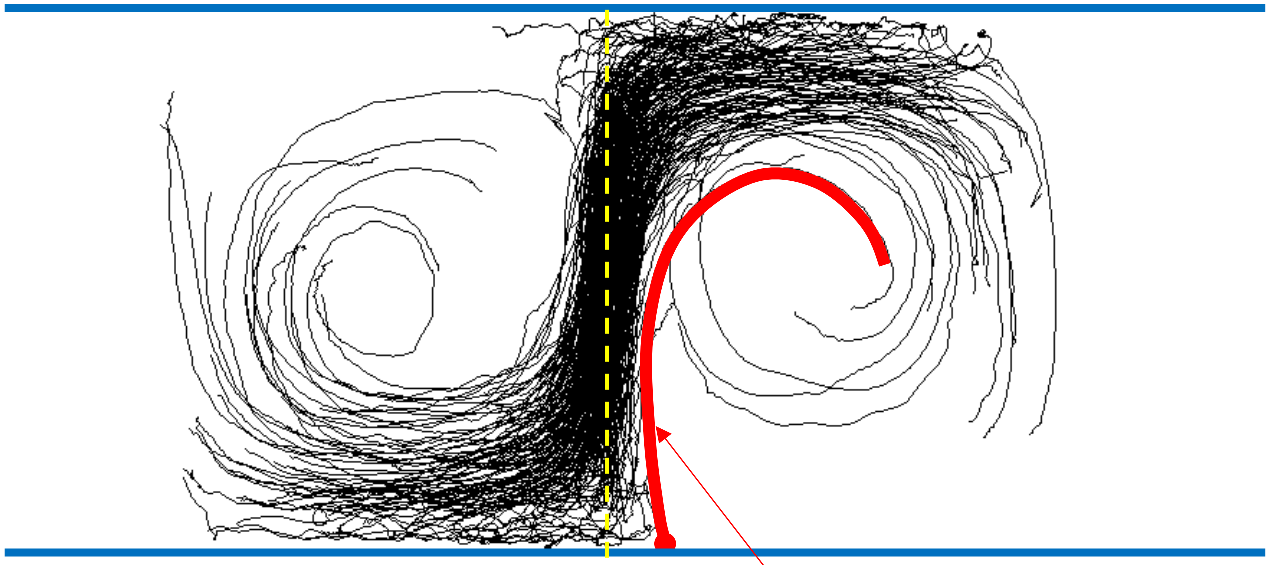


$v_0 = V_s/U = 0.3$



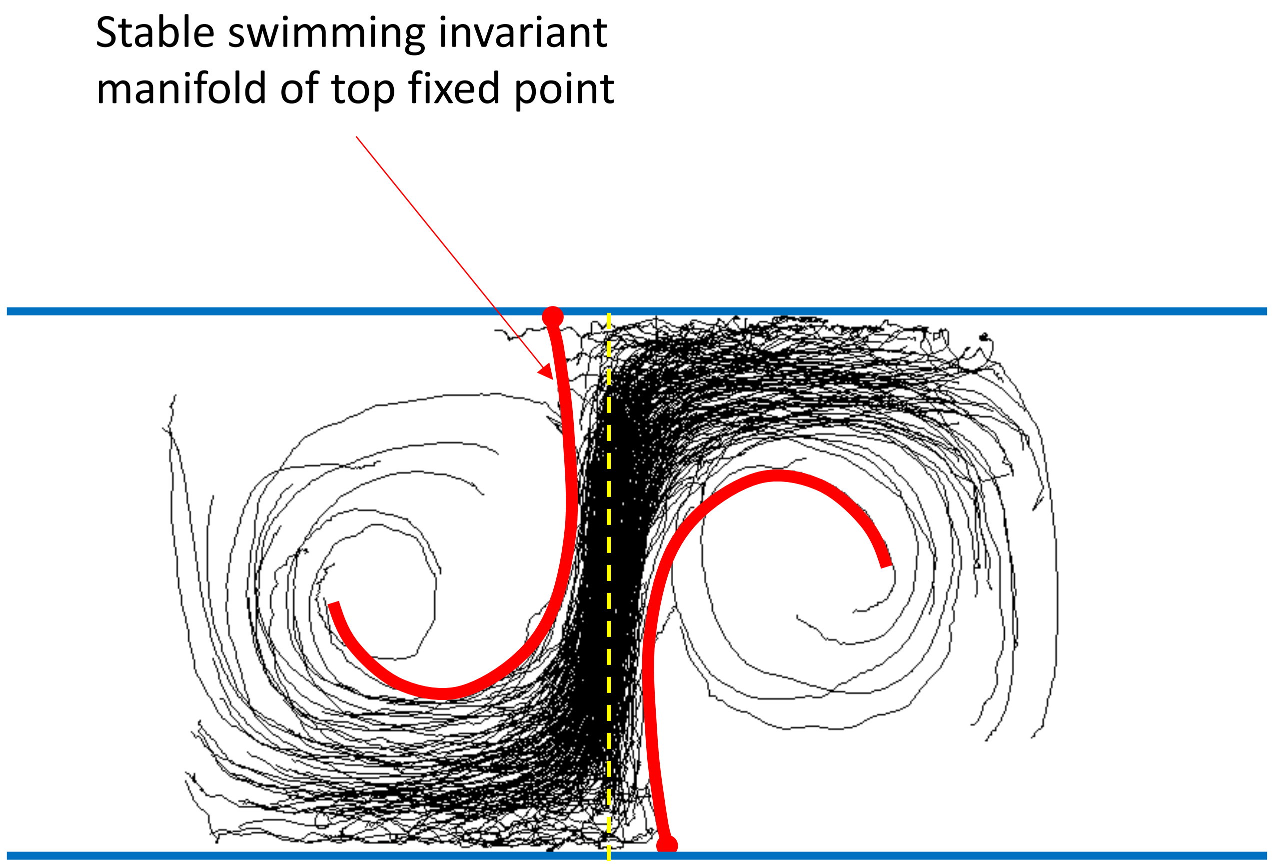
Swimming fixed point (blocking outward swimming)

$v_0 = V_s/U = 0.3$

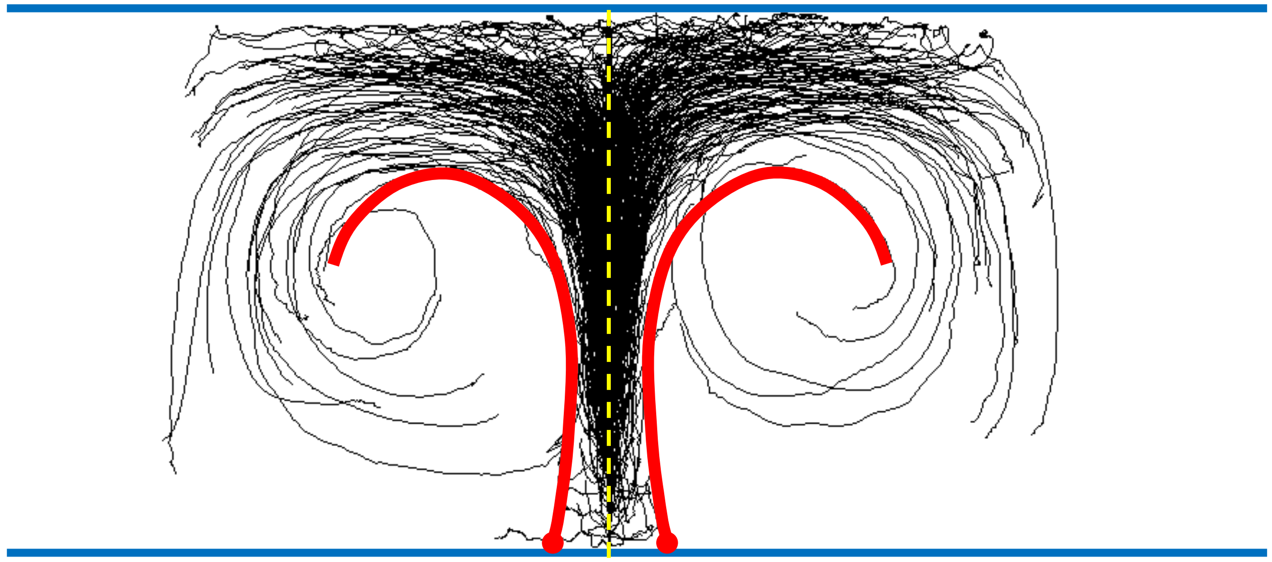


Unstable swimming invariant manifold (blocking outward swimming)

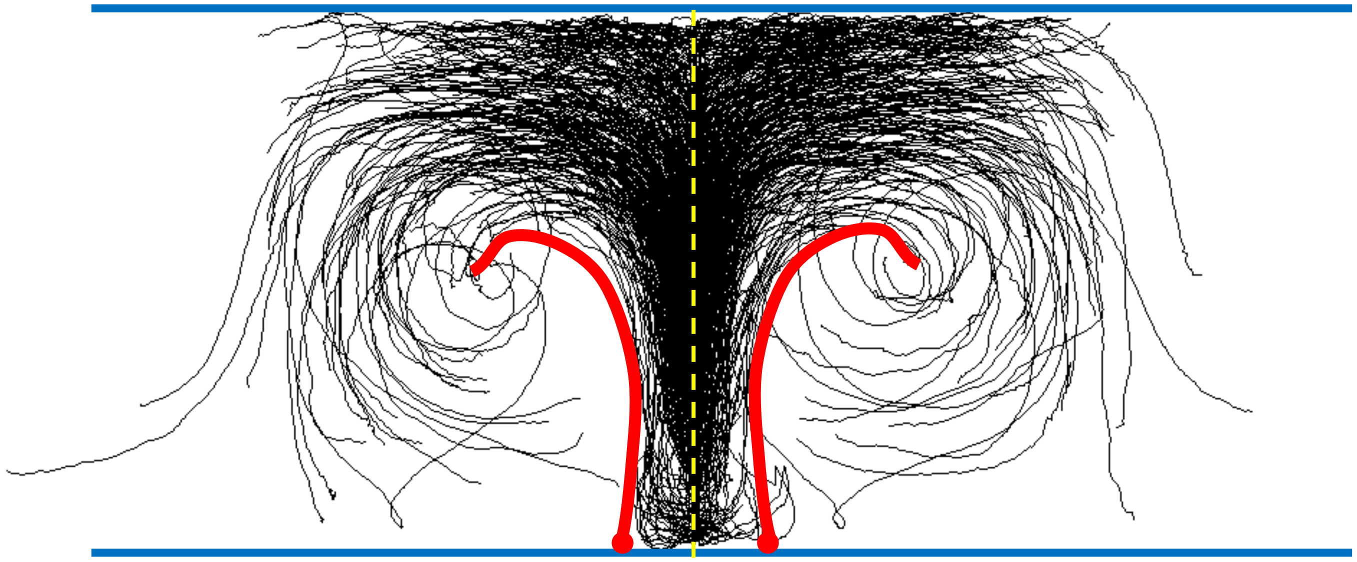
$v_0 = V_s/U = 0.3$



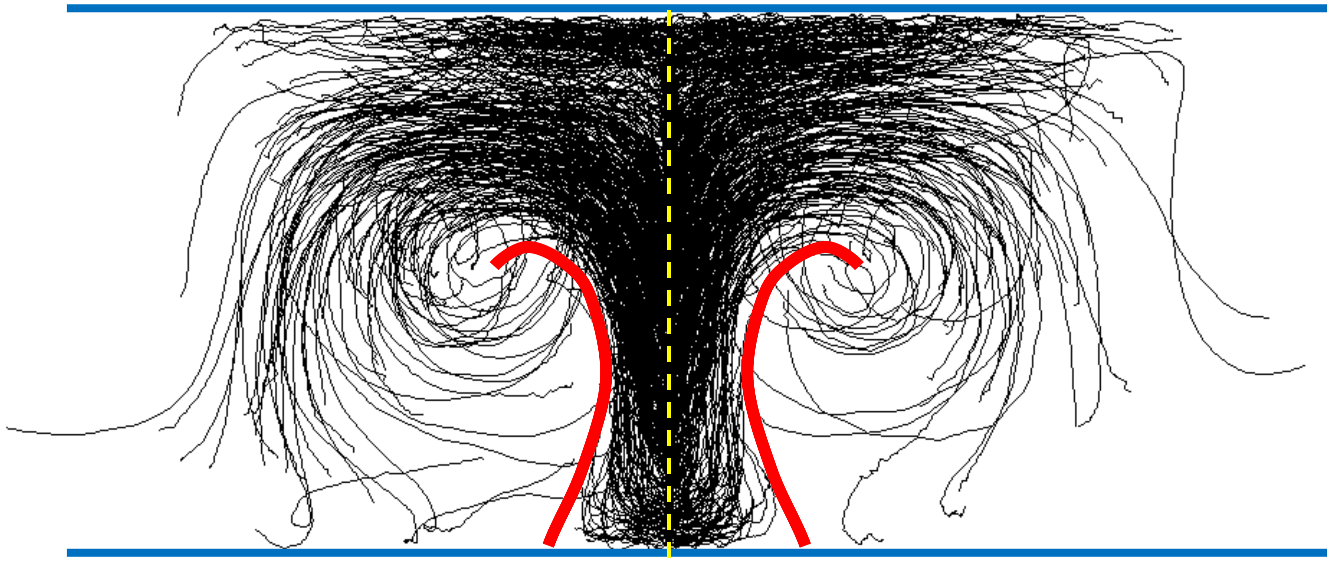
$v_0 = V_s/U = 0.3$



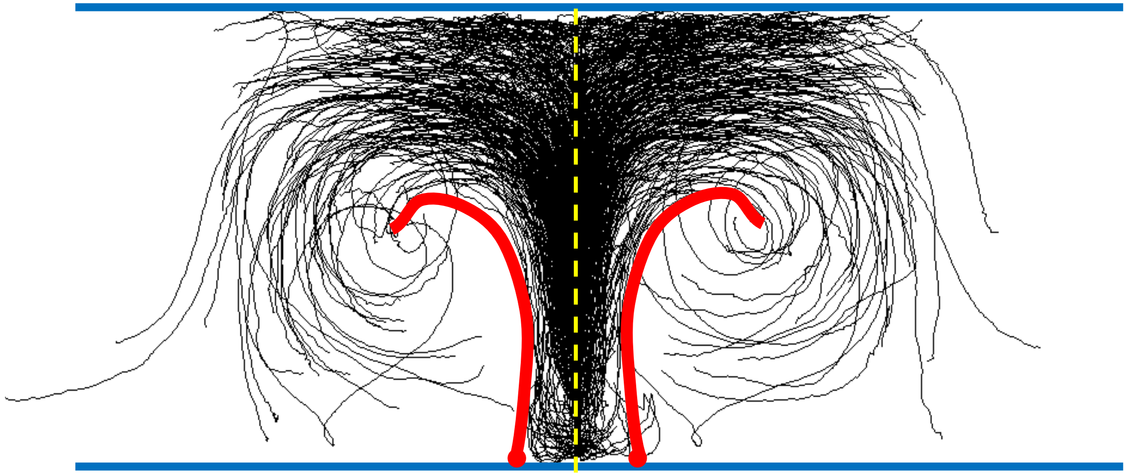
$v_0 = V_s/U = 0.4$



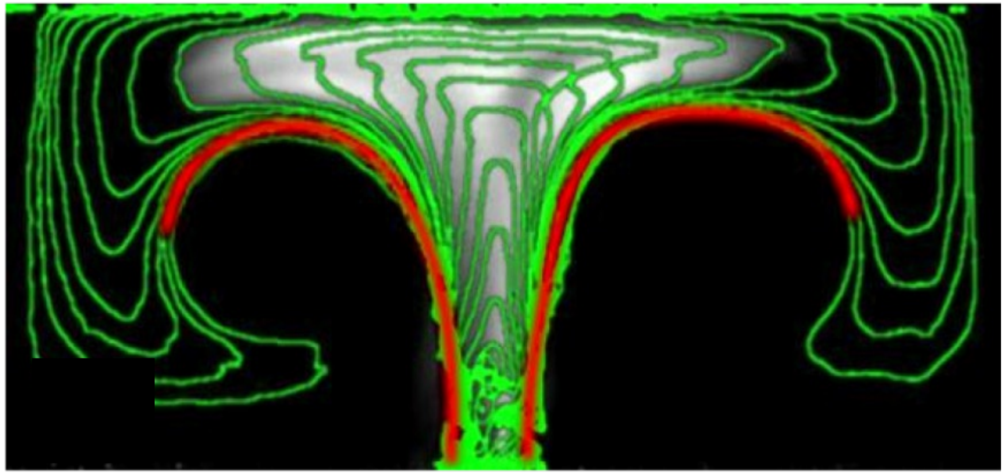
$v_0 = V_s/U = 0.5$



Similar one-way barriers for both swimming microbes and propagating fronts in a vortex-chain flow



Swimming microbes

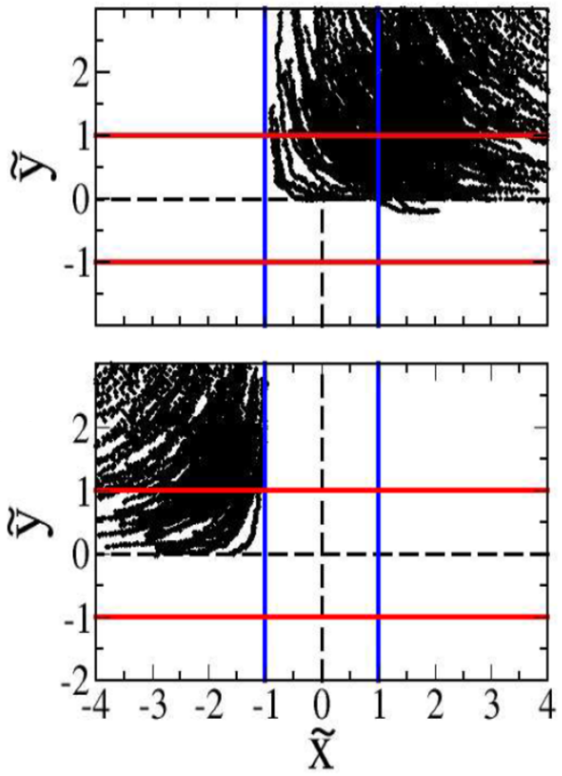


Reaction fronts

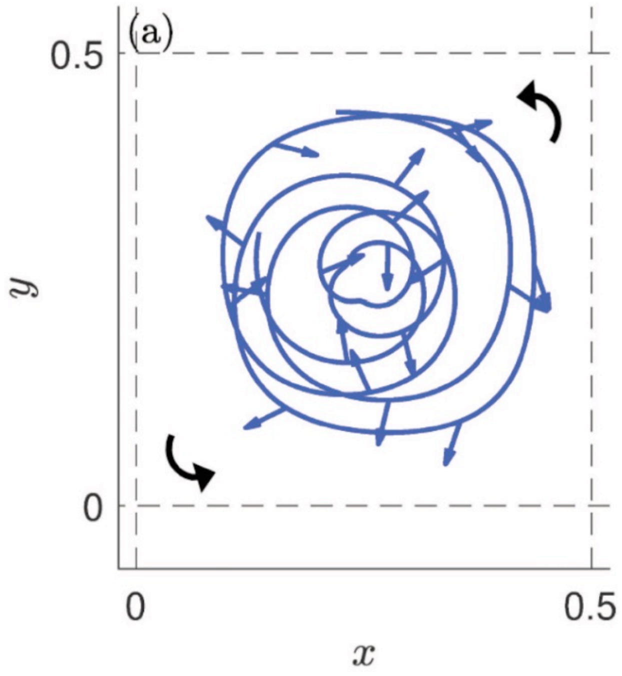
Key takeaway: critical importance of invariant manifolds in at least two, very different active mixing systems.

Swimming Invariant Manifolds have also be measured experimentally for hyperbolic flows.

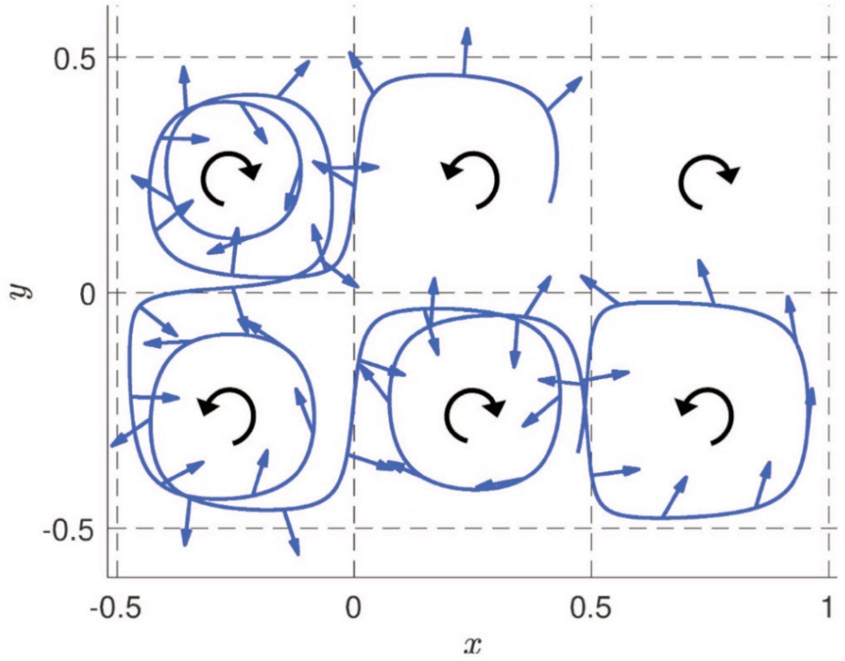
See *Berman, Buggeln, Brantley, Mitchell, and Solomon, Phys. Rev. Fluids 6 L012501 (2021).*



Chaotic and ordered trajectories of active tracers in vortex arrays



Trapped trajectory

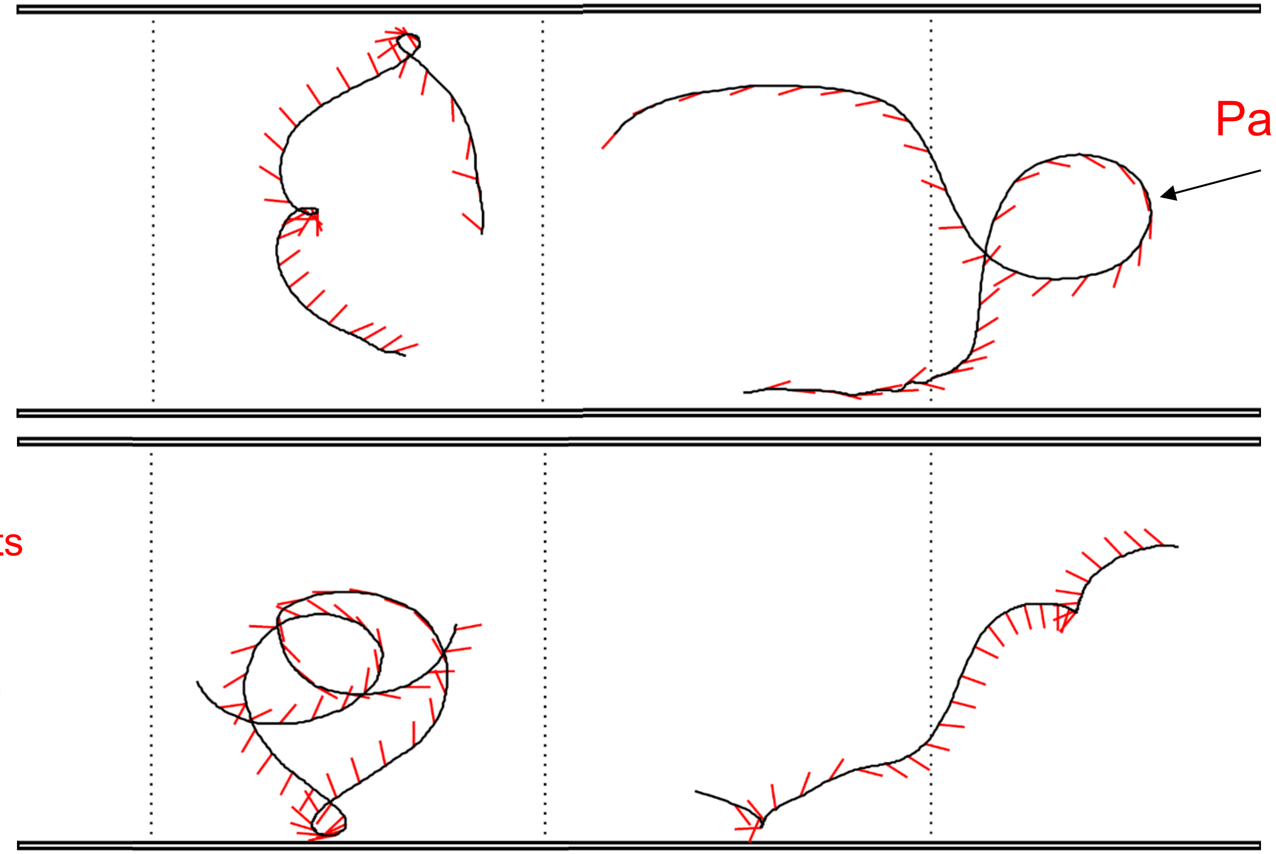


“Escaping” trajectory

Berman & Mitchell, Chaos **30**, 063121 (2020)

Chaotic and ordered trajectories of active tracers in vortex arrays

$v_0 = V_s/U = 0.5$



Part of a chaotic trajectory?

The red segments show the swimming orientation, with the tail at the black curve

Continuing Work

- Evidence of chaotic trajectories and trapping?

  Refs: Khurana, Blawzdziewicz and Ouellette, Phys. Rev. Lett. **106**, 198104 (2011); Berman and Mitchell, Chaos **30**, 063121 (2020).

- Test microbes of different α. (For α = -1, the theory for swimmers is identical to that for reaction fronts.)

- Effects of swimming invariant manifolds on collective behavior of swimmers.

- Effects of chemo- and phototaxis on manifolds.