Correlated variables

Bucknell University PHYS 310

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Error Propagation (old business)

Functional relationship: $Z = f(A, B)$

Measurements:

- $A = \bar{A} \pm \alpha_A$ (a distribution)
- $B = \bar{B} \pm \alpha_B$ (a distribution)

Question: What is the expected distribution of values of Z?
(Specifically, what is the expected standard deviation σ_Z?)

Answer (PHYS 211/212, PHYS 221, PHYS 310, H&H Sect4.2.1):

$$(\alpha_Z)^2 = \left(\alpha_Z^A\right)^2 + \left(\alpha_Z^B\right)^2$$

Question: Where does this come from? Always correct?

Distribution of Z's — the mean

Imagine a large number of A_i's and B_i's sampled from the appropriate distributions. Using the first order term in a Taylor series expansion we get

$$Z_i \simeq f(\bar{A},\bar{B}) + \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} (A_i - \bar{A}) + \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} (B_i - \bar{B})$$

The average value of Z is given by

$$\begin{aligned} \bar{Z} &= f(\bar{A},\bar{B}) + \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} \overline{(A_i - \bar{A})} + \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} \overline{(B_i - \bar{B})} \\ &= f(\bar{A},\bar{B}), \end{aligned} \tag{1}$$

where the quantities like $\overline{(A_i - \bar{A})}$ have averaged to zero (this will happen if the distribution of the values A_i is symmetric around $\bar{A}$).

Distribution of Z's — the sample variance

Reminder:

$$Z_i \simeq f(\bar{A}, \bar{B}) + \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} (A_i - \bar{A}) + \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} (B_i - \bar{B})$$

The sample variance of Z is

$$\begin{aligned}
(\sigma_Z)^2 &= \frac{1}{N-1} \sum_{i=1}^{N} (Z_i - \bar{Z})^2 \\
&= \frac{1}{N-1} \sum \left(\left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} (A_i - \bar{A}) + \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} (B_i - \bar{B}) \right)^2 \\
&= \frac{1}{N-1} \sum \left[\left(\left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} (A_i - \bar{A}) \right)^2 + \left(\left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} (B_i - \bar{B}) \right)^2 \right. \\
&\qquad \left. + 2 \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} (A_i - \bar{A})(B_i - \bar{B}) \right]
\end{aligned}$$

Distribution of Z's — the sample variance

$$
\begin{aligned}
(\sigma_Z)^2 &= \left(\sigma_A \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}}\right)^2 + \left(\sigma_B \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}}\right)^2 \\
&\quad + 2 \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} \frac{1}{N-1} \sum (A_i - \bar{A})(B_i - \bar{B}) \\
&\equiv \left(\sigma_Z^A\right)^2 + \left(\sigma_Z^B\right)^2 + 2 \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} \sigma_{AB},
\end{aligned}
$$

where we have defined the sample *covariance*,

$$
\sigma_{AB} \equiv \frac{1}{N-1} \sum (A_i - \bar{A})(B_i - \bar{B}) \tag{2}
$$

A_i and B_i uncorrelated $\implies \sigma_{AB} = 0$

Distribution of Z's — the standard error

Divde both sides of expression for $(\sigma_Z)^2$ by $N-1$ to get expression in terms of *standard errors*:

$$(\alpha_Z)^2 = \left(\alpha_Z^A\right)^2 + \left(\alpha_Z^B\right)^2 + 2\, \left.\frac{\partial f}{\partial A}\right|_{\bar{A},\bar{B}} \left.\frac{\partial f}{\partial B}\right|_{\bar{A},\bar{B}} \frac{\alpha_{AB}}{N-1}, \qquad (3)$$