Chapter 1

Solving Equations of Motion Using Numerical Iteration

1.1 Introduction

The past few decades have witnessed a massive revolution in the way people live and work, due in great part to *significant* enhancements in computational power. Computers are everywhere these days in society, not just on your desktop (or on your lap) but also in your pockets (MP3 players and cell phones), in your kitchens (ranges, dishwashers and microwave ovens), and behind the scenes monitoring the money in your bank accounts, your class schedules and grades, and your music preferences at on-line music stores.

The significant enhancement in computation power has also dramatically changed all fields of science and engineering. Despite our brilliant teaching of physics in this course, there are many problems in physics and engineering that you simply will not be able to solve analytically.[1] Some problems simply don't allow a closed-form solution. But it is even more severe than that. There are a wide variety of physical systems whose equations of motion **can't** be solved, no matter how brilliant or persistent the scientist/mathematician. In fact, many real systems are "chaotic," with surprisingly complicated behavior arising from seemingly simple systems. In cases where an analytical solution is unavailable, the only option is to solve the problem *numerically*, using a computer to simulate the behavior.

Computer simulations have become among the most important techniques in science and engineering. Many of you will use numerical techniques in your career, whether you are simulating the behavior of a new passenger airline that you are designing, calculating the forces acting on an artificial joint that you are designing for a patient, or predicting the effects of a disruption in Middle East oil supply on the global economy. Numerical sim-

[1] By "analytically" we mean using the tools from mathematics to determine a written solution in the form of an equation that can be used to describe the behavior of the system.

ulations also play a significant role in basic scientific research, enabling us to explore the behavior of a system that is too complicated to solve analytically and too difficult to explore experimentally. In fact, numerical simulations are so common now that they are often considered to be a third branch in scientific analysis, separate from (and complementary to) experimental and theoretical science.

The basic idea of numerical simulations is actually quite easy. In this chapter, we introduce an important technique referred to as *iteration* where we break the dynamics of the system into a series of discrete time steps. So, for example, instead of representing the motion of a ball with a continuous equation, we instead note the location of the ball, say, every tenth of a second. Given the location and velocity of the ball at a particular moment in time, we can predict its location 0.1 s later by using a very simple numerical techniques referred to as the *Euler Method*, a technique that conceptually is nothing more than a simple application of the common "distance = speed $\times$ time" approach. Despite the simplicity of the Euler method, it is a very powerful method that is used in many numerical applications. This chapter introduces the basic ideas (with some homework problems); you will use the method in lab to simulate the motion of a falling object subject to air resistance.

1.2 Solving Newton's second law analytically

Newton's second law $\vec{F}_{\rm net} = m\vec{a}$ is a *differential equation*, i.e., an equation that can be written in terms of derivatives of various quantities. Ideally, we would like to "solve" this differential equation to determine expressions (as a function of time) for the velocity and position of a particle moving under the influence of the forces. If the forces exerted on the particle are all known, then Newton's second law can be rewritten in one-dimension as

$$a_x = \frac{d^2x}{dt^2} = \frac{F_{{\rm net},x}}{m}, \tag{1.1}$$

where the forces are assumed to be possibly functions of position and velocity. Eq. (1.1) written in that form is known as the *equation of motion* for the system under consideration. Mathematically one would proceed by integrating Eq. (1.1) to determine the velocity as a function of time $v_x(t)$ and then integrating once again to obtain the position as a function of time $x(t)$. For example we have learned that for a particle falling from rest from a height x_0 under the force of gravity $F_{\rm net} = mg$, Eq. (1.1) becomes

$$\frac{d^2x}{dt^2} = -g, \tag{1.2}$$

and integrating we obtain the following expressions for the velocity and position:

$$v_x(t) = -gt \qquad \text{and} \qquad x(t) = x_0 - \frac{1}{2}gt^2. \tag{1.3}$$

If you don't understand how we got these expressions, then take the derivative with respect to time of $x(t)$ to get $v_x(t)$ and then $v_x(t)$ to get back to Eq. (1.2).

For the example shown above as well as a few other cases, the equation of motion is relatively straightforward to integrate to get the analytical functions for velocity and position. As discussed in the previous section, though, there are many cases where the equations of motion are not so easy to integrate and other means are necessary for determining the position and velocity of the particle as a function of time.

In the following sections we will develop a set of equations that we can use to calculate the position and velocity of a particle at specified time increments Δt, a technique called *numerical iteration*. Although this technique does not give us as a final result a neat, compact formula for the position and velocity of the particle into which any value of time can be inserted, it does allow us to map out the position and velocity of the particle for an otherwise mathematically intractable problem.

1.3 Numerical Stepping Equations

Let us incorporate the ideas mentioned above into a set of formulas that we (or better yet, a computer) could use to calculate the position and velocity of a particle moving under the influence of some forces. Call the present time t and the time a little later $t + \Delta t$. Let $x(t)$ denote the position of the particle now, then $x(t+\Delta t)$ denotes the position of the particle a short time later. Similarly $v_x(t)$ and $v_x(t + \Delta t)$ represent the present and slightly later velocities of the particle. In all of these expressions, note that $x(t + \Delta t)$ does not mean the quantity 'x' times the quantity '$t + \Delta t$' but rather means the value of the function x evaluated at the time $t + \Delta t$. This is standard functional notation used in mathematics.

Recall the definition of velocity as the rate of change of the position. Taking Δt to be very small in magnitude, we may approximate this as "velocity = displacement/time" and express the velocity at time t approximately as

$$v_x(t) \simeq \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}. \tag{1.4}$$

As you recall, $\Delta x/\Delta t$ is the definition of the average velocity, while the instantaneous velocity is actually the derivative of the position with respect to time. However, for small enough time steps, the average velocity is an excellent approximation for the instantaneous velocity.

Turning the previous expression around, we can write an expression for the position of the particle at time $t+\Delta t$ in terms of the position and velocity at time t:

$$x(t + \Delta t) = x(t) + v_x(t)\Delta t. \tag{1.5}$$

Eq. (1.5) says that the position at time $t + \Delta t$ is the position at time t plus the distance traveled $v_x(t)\Delta t$ by the particle during the short time interval Δt. Notice that this result is only approximate because the velocity v_x at time t is not necessarily equal to the average velocity during the entire time interval. However, if Δt is small enough, the approximation should be quite good.

Next we need an expression for incrementing the velocity. By analogy with the arguments leading up to Eq. (1.5), we can write

$$v_x(t + \Delta t) = v_x(t) + a_x(t)\Delta t. \tag{1.6}$$

The three equations (1.1), (1.5) and (1.6) can now be incorporated into a looping procedure in a computer program. These three equations constitute what is generally referred to as *Euler's method* of numerical approximation. Given an initial position and velocity, we calculate the initial acceleration from Eq. (1.1). Then we calculate the position and velocity a short time later from Eqs. (1.5) and (1.6). Then we repeat the process, pretending that the new values for x and v_x are the initial values. In this way we can numerically iterate the motion of the particle from instant to instant as far into the future as we care to. A spread-sheet program, such as EXCEL, can perform such calculations with very little "programming" required on your part.

A note about numerical errors is worth mentioning. Remember that although Eq. (1.1) is exact, Eqs. (1.5) and (1.6) that update x and v_x to later times are approximations that are best when Δt is small. If the calculations start going haywire, we can help the situation by choosing smaller steps. This means of course that the computer will have to run longer, but that's frequently not a serious problem.

1.4 Numerical Solution for a Mass on a Spring

Let's apply this new method to a system we will be studying more in depth later in this course. The system is a mass which moves under the influence of a force exerted on it by a spring. The spring is a device which exerts a force which is proportional to the displacement of the mass from an equilibrium position. Taking the equilibrium position to be $x = 0$, this implies that the acceleration of the mass is directly proportional to the position $x(t)$. Suppose in our particular system the acceleration is given by

$$a_x(t) = -2.00\, x(t). \tag{1.7}$$

The minus sign in this expression tells us that the force is always opposite to the displacement. We'll also assume that time is in seconds, position is in meters, velocity is in meters per second, and acceleration is in meters per second squared

To proceed, we choose time steps of size $\Delta t = 0.10$ s and start the clock at $t = 0$. We could pick any initial position and velocity; let's choose to release the mass from rest at a position 0.30 m from equilibrium, i.e. $x(0) = 0.30$ m and $v_x(0) = 0$. Let's walk through the first few steps and then show some results from a computer spreadsheet.

For our example Eqs. (1.1), (1.5) and (1.6) are written as

$$a_x(t) = -2.00\, x(t) \tag{1.8}$$

$$x(t + 0.10) = x(t) + 0.10\, v_x(t) \tag{1.9}$$

$$v_x(t + 0.10) = v_x(t) + 0.10\, a_x(t). \tag{1.10}$$

First calculate the initial acceleration by setting $t = 0$ in Eq. (1.8) to find

$$a_x(0) = -2.00\, x(0) = -2.00 \times 0.30 = -0.60. \tag{1.11}$$

Then update $x(t)$ and $v_x(t)$ by setting $t = 0$ in Eqs. (1.9) and (1.10):

$$x(0.10) = x(0) + 0.10\, v_x(0) = 0.30 + 0.10 \times 0 = 0.30 \tag{1.12}$$

$$v_x(0.10) = v_x(0) + 0.10\, a_x(0) = 0 + 0.10 \times (-0.60) = -0.06. \tag{1.13}$$

Since the mass was initially at rest, a short time later it is still approximately at the same location. However, since the spring is stretched at $t = 0$, a force is acting on the mass immediately, so that a short time later it has already acquired a non-zero velocity.

How would you find $x(0.20)$ and $v_x(0.20)$? Again use Eqs. (1.8) through (1.10), this time with the 'present time' $t = 0.10$. We find that

$$a_x(0.10) = -2.00\, x(0.10) = -2.00 \times 0.30 = -0.60 \tag{1.14}$$

$$\begin{aligned} x(0.20) &= x(0.10) + 0.10\, v_x(0.10) \\ &= 0.30 + 0.10 \times (-0.06) \\ &= 0.294 \end{aligned} \tag{1.15}$$

$$\begin{aligned} v_x(0.20) &= v_x(0.10) + 0.10\, a_x(0.10) \\ &= -0.06 + 0.10 \times (-0.60) \\ &= -0.12. \end{aligned} \tag{1.16}$$

We can continue this process as long as we like. You will find it convenient to organize the information for the position, velocity and acceleration for each time in the form of a table. Table 1.1 on the next page lists t, x, v_x and a_x for the motion of this mass. Note that the periodic nature of the motion is manifested in the entries of the table.

There is an unsettling aspect of the entries in Table 1.1. We started at $x = 0.30\,\text{m}$, but at $t = 2.30\,\text{s}$ the position of the mass is $x = -0.375\,\text{m}$, and further down in the table we find that at $t = 4.5$ s, $x = 0.468\,\text{m}$. What should we have expected? If we had a real mass connected to a spring and set it oscillating we would expect the amplitude of the oscillations to gradually decrease because of the presence of dissipative forces (air resistance and the imperfect elasticity of the spring). In an ideal case, with no dissipative effects, we would expect there to be no increase or decrease in the amplitude; that is, the mass should oscillate between $x = +0.30\,\text{m}$ and $x = -0.30\,\text{m}$. But this is not the case if we look at the data in Table 1.1. The problem is that we used too large a time increment. Why does too large a time increment lead to errors? If you recall, our stepping equations use the approximation that the average velocity is very close to the instantaneous velocity. If the time step is too large, this approximation is no longer valid and leads to errors.

We can improve our calculation of the motion by choosing a smaller time increment Δt. If we choose $\Delta t = 0.01\,\text{s}$ rather than $0.10\,\text{s}$, we would be calculating over a much finer time interval (10 times smaller) and while we will have to do 10 times more computations to evolve the motion out to the same time, the calculations should be more accurate. Table 1.2 lists t, x, v_x and a_x near a point of maximum displacement for this smaller time increment. The maximum displacement is now about 0.314. This is still larger than the initial displacement but not nearly as bad as before. Further reduction of the time increment would improve the result.

Table 1.1: Numerical solution for motion of mass on a spring using $\Delta t = 0.10\,\mathrm{s}$

t	$x(t)$	$v_x(t)$	$a_x(t)$
0	0.300	0	-0.600
0.100	0.300	-0.060	-0.600
0.200	0.294	-0.120	-0.588
0.300	0.282	-0.179	-0.564
0.400	0.264	-0.235	-0.528
0.500	0.241	-0.288	-0.481
0.600	0.212	-0.336	-0.424
0.700	0.178	-0.379	-0.356
0.800	0.140	-0.414	-0.281
0.900	0.099	-0.442	-0.198
1.000	0.055	-0.462	-0.109
1.100	0.008	-0.473	-0.017
1.200	-0.039	-0.475	0.078
1.300	-0.086	-0.467	0.173
1.400	-0.133	-0.450	0.266
1.500	-0.178	-0.423	0.356
1.600	-0.220	-0.387	0.440
1.700	-0.259	-0.343	0.518
1.800	-0.293	-0.292	0.587
1.900	-0.322	-0.233	0.645
2.000	-0.346	-0.168	0.692
2.100	-0.363	-0.099	0.725
2.200	-0.373	-0.027	0.745
2.300	-0.375	0.048	0.750
2.400	-0.370	0.123	0.741
2.500	-0.358	0.197	0.716
2.600	-0.338	0.268	0.677
2.700	-0.312	0.336	0.623
2.800	-0.278	0.398	0.556
2.900	-0.238	0.454	0.476
3.000	-0.193	0.502	0.386
3.100	-0.143	0.540	0.285
3.200	-0.089	0.569	0.177
3.300	-0.032	0.587	0.063
3.400	0.027	0.593	-0.054
3.500	0.086	0.587	-0.173
3.600	0.145	0.570	-0.290
3.700	0.202	0.541	-0.404
3.800	0.256	0.501	-0.512
3.900	0.306	0.450	-0.612
4.000	0.351	0.388	-0.702
4.100	0.390	0.318	-0.780
4.200	0.422	0.240	-0.844
4.300	0.446	0.156	-0.892
4.400	0.461	0.067	-0.923
4.500	0.468	-0.026	-0.936
4.600	0.465	-0.119	-0.931

Table 1.2: Data for mass on a spring near a turning point using $\Delta t = 0.01\,\mathrm{s}$

t	$x(t)$	$v_x(t)$	$a_x(t)$
4.190	0.293	0.155	-0.586
4.200	0.295	0.149	-0.589
4.210	0.296	0.143	-0.592
4.220	0.297	0.137	-0.595
4.230	0.299	0.131	-0.598
4.240	0.300	0.125	-0.600
4.250	0.301	0.119	-0.603
4.260	0.303	0.113	-0.605
4.270	0.304	0.107	-0.607
4.280	0.305	0.101	-0.610
4.290	0.306	0.095	-0.612
4.300	0.307	0.089	-0.614
4.310	0.308	0.083	-0.615
4.320	0.308	0.077	-0.617
4.330	0.309	0.071	-0.619
4.340	0.310	0.064	-0.620
4.350	0.311	0.058	-0.621
4.360	0.311	0.052	-0.622
4.370	0.312	0.046	-0.623
4.380	0.312	0.040	-0.624
4.390	0.313	0.033	-0.625
4.400	0.313	0.027	-0.626
4.410	0.313	0.021	-0.626
4.420	0.313	0.014	-0.627
4.430	0.314	0.008	-0.627
4.440	0.314	0.002	-0.627
4.450	0.314	-0.004	-0.627
4.460	0.314	-0.011	-0.627
4.470	0.313	-0.017	-0.627
4.480	0.313	-0.023	-0.627
4.490	0.313	-0.029	-0.626
4.500	0.313	-0.036	-0.626
4.510	0.312	-0.042	-0.625
4.520	0.312	-0.048	-0.624
4.530	0.312	-0.054	-0.623
4.540	0.311	-0.061	-0.622
4.550	0.310	-0.067	-0.621
4.560	0.310	-0.073	-0.619
4.570	0.309	-0.079	-0.618
4.580	0.308	-0.085	-0.616
4.590	0.307	-0.092	-0.615
4.600	0.306	-0.098	-0.613
4.610	0.305	-0.104	-0.611
4.620	0.304	-0.110	-0.609
4.630	0.303	-0.116	-0.607
4.640	0.302	-0.122	-0.604
4.650	0.301	-0.128	-0.602

Problems

1. For a certain mass-spring system the acceleration is given by $a_x(t) = -0.10\,x(t)$. Suppose the initial position and velocity are $x(0) = 10$ m and $v_x(0) = -1.0\,\mathrm{m/s}$. Calculate $x(t)$ and $v_x(t)$ at $t = 2\,\mathrm{s}$ in two different ways:

(a) Use two steps of 1 second each.

(b) Use four steps of $\frac{1}{2}$ second each. Round only your final results to three digits (keep all digits for the intermediate calculations).

(c) Why aren't the answers to a) and b) the same?

2. A drag force on an object is opposite to its velocity and is often proportional to its speed. Let's immerse the mass-spring system of problem (1) in a vat of salad oil so that the acceleration becomes

$$a_x(t) = -0.10\,x(t) - v_x(t)$$

Repeat problem 1.1 for this acceleration. Compare the results with those you originally got in problem 1.1. Are the results what you might expect when a drag force is present?