Appendix A: Notes on Propagation of Uncertainties and Statistics

A.1 Propagation of Uncertainties

Often one must compute a value of some physical quantity from measurements of two or more physical quantities. For example, the gravitational field, g, at the surface of the Earth can be computed from measurements of the length L and period of oscillation T of a small amplitude pendulum, using the theoretically derived formula

$$g = g(L, T) = \frac{4\pi^2 L}{T^2}. \tag{A.1}$$

However, the measurements of both L and T are subject to experimental errors, perhaps better called uncertainties, and the question we want to answer here is, "What is the uncertainty in our computed value of g?" There is uncertainty in g due to uncertainty in L, and uncertainty in g due to uncertainty in T. These uncertainties could be due to the finite size of the smallest scale divisions (on the meter stick or clock) or to the fact that multiple readings of an indicating device are not all the same, or to the fact that a needle on a scale fluctuates as one tries to read it. For example, suppose L has been measured to be $0.96\,\mathrm{m}$ with an uncertainty of $\Delta L = 0.01\,\mathrm{m}$, and T has been measured to be $1.97\,\mathrm{s}$ with an uncertainty of $\Delta T = 0.02\,\mathrm{s}$. The uncertainty in g due to uncertainty in L, which we call Δg_L, can be computed by calculating g twice, once using the length $0.96\,\mathrm{m}$ and again using the length $0.97\,\mathrm{m}$ (or $0.95\,\mathrm{m}$) and subtracting. Thus,

$$\begin{aligned} \Delta g_L &= g(L + \Delta L, T) - g(L, T) \\ &= \frac{4\pi^2 \times 0.97}{1.97^2} - \frac{4\pi^2 \times 0.96}{1.97^2} \\ &= 0.10\,\mathrm{m/s^2}. \end{aligned} \tag{A.2}$$

Similarly, the uncertainty in g due to uncertainty in T, which we call Δg_T, is computed by calculating g twice, once using the period 1.97 s and again using the period 1.99 s (or 1.95 s) and subtracting. Thus,

$$\begin{aligned} \Delta g_T &= g(L, T + \Delta T) - g(L, T) \\ &= \frac{4\pi^2 \times 0.96}{1.99^2} - \frac{4\pi^2 \times 0.96}{1.97^2} \\ &= -0.20\,\text{m/s}^2. \end{aligned} \tag{A.3}$$

The total uncertainty in g is a combination of Δg_L and Δg_T:

$$\Delta g = \sqrt{(\Delta g_L)^2 + (\Delta g_T)^2}. \tag{A.4}$$

In our case, $g = 4\pi^2 L/T^2 = 4\pi^2 \times 0.96/1.97^2 = 9.77\,\text{m/s}^2$, the uncertainty is $\Delta g = \sqrt{(-0.20)^2 + (0.10)^2} = 0.22\,\text{m/s}^2$, and the fractional uncertainty in g is

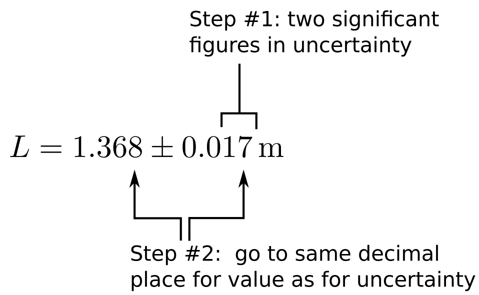
$$\frac{\Delta g}{g} = \frac{0.22}{9.77} = 0.02 = 2\%. \tag{A.5}$$

When multiple measurements are made, the uncertainties L and T may be *standard deviations of the mean* of the measurements (see the next section of this appendix) rather than simply estimates of the uncertainties. This does not alter the validity of the method of calculating Δg_L and Δg_T or the method of combining them to obtain Δg, the overall uncertainty in g.

A.2 Reporting a numerical result

Significant figures are important because they convey the accuracy of data, and this information is key when interpreting that data for future use. In general, the *uncertainty* of a final answer should be rounded to one or two significant figures (why not more?). Then the answer itself should be rounded so that its least significant figure is in the same decimal place as the least significant figure of the uncertainty. This is an important step; it would be confusing to say that you knew a value accurately to the hundredths place when your uncertainty states that you are unsure about the tenths place of the value.

For example, if the final value for the measurement of the length of an object, L, is 1.3679625 m and the uncertainty of that answer is 0.01684 m, the final result should be stated as:



For answers involving scientific notation, the following format is acceptable and recommended:

$$L = (1.612 \pm 0.041) \times 10^{-11}\,\text{m}. \tag{A.6}$$

Another common format for the same data is the following:

$$L = 1.612(41) \times 10^{-11}\,\text{m}. \tag{A.7}$$

In this format the value in parentheses, (41), is the uncertainty, and the decimal place value associated with the uncertainty is the same as that of the final two numbers in the value of L. The uncertainties in the fundamental constants listed on the inside back cover of this manual are given in this format.

A.3 Statistics

The universe of possible measurements

Imagine making a very large number of measurements of a physical parameter, say x. We'll call this set of measurements the *universe of possible measurements*. Time prevents you from actually making all these measurements; instead you usually perform only a small subset of them. The measurements in the universe of possible measurements are distributed around some mean value, called the *true mean*, typically in the manner illustrated in Fig. A.1. Here, most of the measurements lie relatively near the true mean, as shown by the peak in the distribution curve, but a few lie at much larger or smaller values.

The true mean μ is given by

$$\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i, \tag{A.8}$$

where the number of measurements N approaches infinity (because the universe of possible measurements is large).

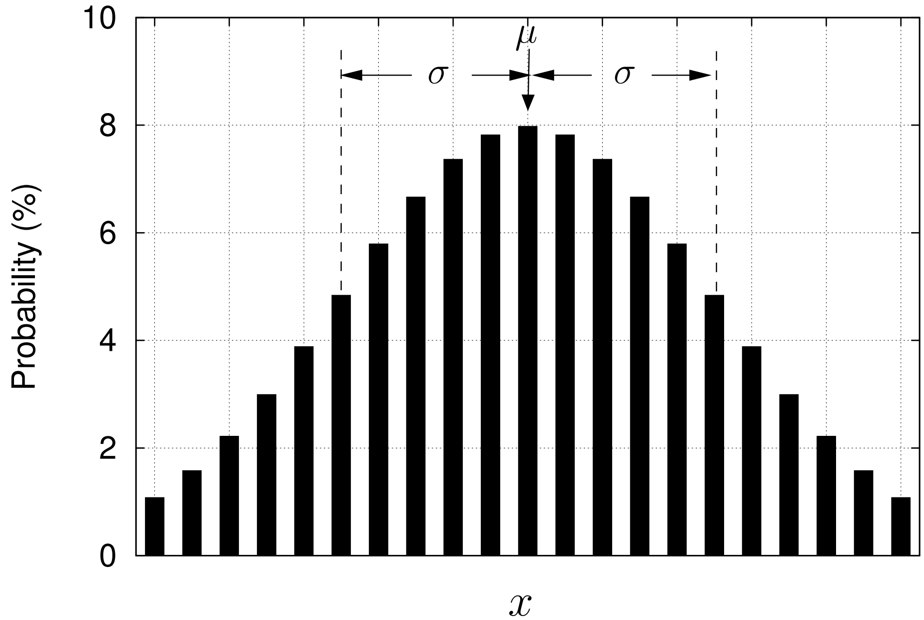


Figure A.1: Probability distribution of *universe of possible measurements*.

The width of the distribution is characterized by a parameter called the *true standard deviation*, σ:

$$\sigma = \lim_{N\to\infty} \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}. \tag{A.9}$$

When the distribution has the typical Gaussian shape of Fig. A.1, approximately 68% of the measurements lie within $\pm\sigma$ of the true mean, 95% within $\pm 2\sigma$ of the true mean, and 99.74% within $\pm 3\sigma$ of the true mean. In lab, we will consider two values consistent with each other if one falls within $\pm 2\sigma$ of the other.

Your job as an experimenter is to estimate the true mean and the true standard deviation from just a few measurements. If you make just one measurement, that measurement has a 68% chance of lying within $\pm\sigma$ of the true mean. But this statement tells you nothing because neither the true mean nor the true standard deviation is known ahead of time. In fact, a single measurement, while giving a ball park estimate of the true mean, says nothing at all about the true standard deviation. It is for this reason that, almost always, you must make multiple measurements.

Estimates of the true mean and true standard deviation

The best estimates of μ and σ that can be obtained from the finite set of N measurements you actually make are given by the experimental mean and standard deviation. The experimental mean and standard deviation are given by the same formulas that you have seen previously, without the limit $N \to \infty$.

$$\text{experimental mean} = \langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i, \tag{A.10}$$

$$\begin{array}{c}\text{experimental}\\ \text{standard deviation}\end{array} = s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \langle x \rangle\right)^2} = \sqrt{\frac{N}{N-1}\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle}. \tag{A.11}$$

An equivalent version of Eq. (A.11) is

$$s = \sqrt{\frac{N}{N-1}\left(\langle x^2 \rangle - \langle x \rangle^2\right)}. \tag{A.12}$$

You should be able to show that this is true (i.e., show that $\left\langle \left(x - \langle x \rangle\right)^2 \right\rangle = \langle x^2 \rangle - \langle x \rangle^2$). (Hint: Expand the last expression on the left-hand side of Eq. (A.11).) Equation (A.12) is frequently more convenient for calculational purposes. The first term under the square root sign is the average of the squares of the x's; that is, each x is squared, these squares are added, and the sum is divided by N. Nearly all calculators have a way of adding a number to memory so that each x and x^2 can be summed without entering x twice. This can save considerable button pushing when the measurements are several digits long. On the other hand, round-off errors in the numerical calculation of s are more likely to be important in Eq. (A.12) than in Eq. (A.11). (Why?) To determine the experimental standard deviation in Excel use `STDEV()`.

Standard deviation of the mean

While the standard deviation s tells you how close an individual measurement is likely to be to the true mean, we would like to know how far $\langle x \rangle$ is from the true mean. Because $\langle x \rangle$ is an average of N measurements, and is calculated from measurements that lie both above and below the true mean, you might correctly guess that $\langle x \rangle$ is likely to lie closer to the true mean than a typical individual measurement. But how much closer? The answer turns out to be that $\langle x \rangle$ has a 68% chance of lying within $\pm s/\sqrt{N}$ of the true mean; or, equivalently, the true mean has a 68% chance of lying within $\pm s/\sqrt{N}$ of $\langle x \rangle$. Here N, as in Eq. (A.10), is the number of values used to obtain the mean.

The uncertainty of the mean, given by $s/\sqrt{N}$, is called the *standard deviation of the mean.* This is an accurate name, for $s/\sqrt{N}$ is the best estimate of what you would get if you measured the mean of N measurements many times and then computed the experimental standard deviation of these means.

So, whenever you quote the uncertainty of a quantity that you've measured N times, you should quote the standard deviation of the mean,

$$\text{uncertainty} = \text{standard deviation of the mean} = \frac{s}{\sqrt{N}} \tag{A.13}$$