ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Friday (3:00 pm lecture), Sept. 18, 2026

1. Symmetric matrices ($A^T = A$) behave well
   a. Eigenvalues are real; all eigenvectors are linearly independent (LI)
   b. Distinct eigenvalues → orthogonal eigenvectors (also LI)
   c. Example of above (Verify with *Matlab* using `eig` function):

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & -2 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

   d. Repeated eigenvalues → LI eigenvectors, but they might not be orthogonal
   e. Linearly independent ≠ orthogonal, but all orthogonal matrices (defined below) have LI eigenvectors
   f. Symmetric matrices can be singular and therefore have at least one zero eigenvalue; even so, all eigenvectors are LI.
   g. Although the eigenvectors of symmetric matrices are all LI, a nonsingular symmetric matrix could still be ill-conditioned
   h. Extreme example of case "e" above: Find the eigenvalues and eigenvectors of matrix A below:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   By inspection, the eigenvalues are all zero; that is, $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Now, find the first eigenvector:

$$\left(A - \lambda_1 I\right)\mathbf{x}_1 = \mathbf{0} \quad \rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x}_1 = \mathbf{0}$$

   We can choose any vector to be the first eigenvector $\mathbf{x}_1$. Likewise, we can choose any vectors to be the eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$. We can therefore choose linearly independent eigenvectors. (Verify with *Matlab* using `eig` function.)

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2. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)
    a. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each column vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
    b. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because such a matrix satisfies $A^{-1} = A^T = A$.)
    c. Example: Check that $A^{-1} = A^T$ and that each column is normalized

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

3. Where we are heading: *LU* and *QR* factorizations and the SVD