ENGR 4/695 Advanced Topics in Engineering Mathematics Fall 2026

Lecture Outline for Monday, Sept. 14

1. Introduction to eigenvalues, eigenvectors, and eigensystems:

   a. Consider the system $A\mathbf{x} = \mathbf{b}$, where A is an $N \times N$ matrix. It was noticed a long time ago that for some nonzero vectors $\mathbf{b}$, the solution vector $\mathbf{x}$ points in the same direction, although $\mathbf{x}$ usually (but not always) has a different length than $\mathbf{b}$ so that $\mathbf{b} = \lambda\mathbf{x}$, where λ is a scalar.

   b. Those vectors are called *eigenvectors*, and the scalar multipliers are called *eigenvalues*.

   c. An $N \times N$ matrix has N eigenvalues and N eigenvectors (which may or may not be distinct) that satisfy the relationship

   $$A\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad \rightarrow \quad \left(A - \lambda_i I\right)\mathbf{x}_i = \mathbf{0}, \qquad i = 1, 2, 3, \ldots, N,$$

   d. The relationship above can be used to find the eigenvalues and eigenvectors.

   e. It turns out that eigenvalues and eigenvectors have a lot of useful applications and can often provide insight. They can also tell us a lot about the matrix A itself and the trustworthiness or accuracy of the computed solution to the system $A\mathbf{x} = \mathbf{b}$.

2. First, some examples: Compute the eigenvalues and eigenvectors of the matrices below:

   $$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 4 \\ 0 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

   a. Theorem 8.6.6 in the textbook (paraphrased): "A homogeneous $N \times N$ system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ if and only if A is singular." (*Homogeneous* means that the right-hand-side vector is $\mathbf{0}$, and *nontrivial* means that $\mathbf{x}$ is not a zero vector.)

   b. If nontrivial eigenvectors of A exist, then the matrix expression $A - \lambda\mathbf{x}$ must be singular. It therefore has a determinant of zero, so the eigenvalues of A can be found as shown below.

   $$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad \rightarrow \quad \left(1-\lambda\right)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$$

   or

   $$\left(\lambda - 3\right)\left(\lambda + 1\right) = 0 \quad \rightarrow \quad \lambda = 3, -1$$

   This is called the *characteristic equation* of A. The solutions are the eigenvalues. There are two eigenvalues because A is a 2×2 matrix ($N = 2$).

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c. To calculate the eigenvectors, solve the system $(A - \lambda_i \mathbf{x}_i) = \mathbf{0}$ for $i = 1$ and 2. Start with $i = 1$, for which $\lambda_1 = 3$:

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Note that $A - \lambda_1 \mathbf{x}_1$ has a rank of 1, which means that it is singular because the rank is less than $N = 2$. This is to be expected in light of Theorem 8.6.6. The system $(A - \lambda_1 \mathbf{x}_1) = \mathbf{0}$ has infinitely many solutions, but the solutions must satisfy the relationship

$$-2x_1 + 2x_2 = 0 \quad \rightarrow \quad x_1 = x_2$$

Any vector with elements that satisfy the relationship above is an eigenvector associated with $\lambda_1 = 3$. Possible eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -170 \\ -170 \end{bmatrix}$$

Eigenvectors are not unique because they can have any length. However, they usually point in a unique direction (or in the directly opposite direction if scaled by a negative factor).

d. Eigenvector associated with $i = 2$, for which $\lambda_2 = -1$:

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Again, $A - \lambda_1 \mathbf{x}_1$ has a rank of 1, which means that it is singular. The elements of the eigenvector must satisfy the relationship

$$2x_1 + 2x_2 = 0 \quad \rightarrow \quad x_1 = -x_2$$

Possible eigenvectors associated with $\lambda_2 = -1$ are

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 570 \\ -570 \end{bmatrix}$$

Note that the eigenvectors of matrix A happen to be orthogonal. However, not *all* eigenvectors are orthogonal.

e. Now find the eigenvalues and eigenvectors of B and C. You can check your answers with *Matlab*.

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3. The eigenvectors of the inverse of a nonsingular matrix are the same as those of the original matrix, and the eigenvalues of the inverse are the reciprocal of those of the original. A quick proof:

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i$$
$$A^{-1}\left(A\mathbf{x}_i\right) = A^{-1}\left(\lambda_i \mathbf{x}_i\right)$$
$$\left(A^{-1}A\right)\mathbf{x}_i = \lambda_i A^{-1}\mathbf{x}_i$$
$$\mathbf{x}_i = \lambda_i A^{-1}\mathbf{x}_i$$
$$A^{-1}\mathbf{x}_i = \frac{1}{\lambda_i}\mathbf{x}_i$$

4. Calculate the eigenvalues and eigenvectors of A^{-1}, and compare them to those of the original matrix A above. The inverse of A is given below:

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$