PHYSICS 212

SUPPLEMENTARY READING

Spring 2025

Fundamental Constants

2018 CODATA values from NIST, at `http://physics.nist.gov/cuu/Constants`

Numbers in parentheses are uncertainties in the last two digits of the listed value.

speed of light in vacuum	$c = 299\ 792\ 458\,\text{m/s}$ (exact)	
Planck constant	$h = 6.626\ 070\ 15 \times 10^{-34}\,\text{J}\cdot\text{s}$ (exact)	
	$\hbar = 1.054\ 571\ 818\ \cdots \times 10^{-34}\,\text{J}\cdot\text{s}$	
	$hc = 1240 \cdots\ \text{eV}\cdot\text{nm}$	
constant of gravitation	$G = 6.674\ 30(15) \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$	
Coulomb force constant	$k_\text{C} = 8.987\ 551\ 792\ 2(13) \times 10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2$	
permittivity constant	$\epsilon_0 = 8.854\ 187\ 812\ 8(13) \times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2$	
permeability constant	$\mu_0 = 4\pi \times 1.000\ 000\ 000\ 54(15) \times 10^{-7}\,\text{N/A}^2$	
elementary charge	$e = 1.602\ 176\ 634 \times 10^{-19}\,\text{C}$ (exact)	
Avogadro constant	$N_A = 6.022\ 140\ 76 \times 10^{23}$ molecules/mol (exact)	
Boltzmann constant	$k_B = 1.380\ 649 \times 10^{-23}\,\text{J/K}$ (exact)	$\simeq 8.62 \times 10^{-5}\,\text{eV/K}$
electron mass	$m_e = 9.109\ 383\ 7015(28) \times 10^{-31}\,\text{kg}$	$\simeq 0.511\,\text{MeV}/c^2$
proton mass	$m_p = 1.672\ 621\ 923\ 69(51) \times 10^{-27}\,\text{kg}$	$\simeq 938.3\,\text{MeV}/c^2$
neutron mass	$m_n = 1.674\ 927\ 498\ 04(95) \times 10^{-27}\,\text{kg}$	$\simeq 939.6\,\text{MeV}/c^2$
electron magnetic moment	$\mu_e = -9.284\ 764\ 7043(28) \times 10^{-24}\,\text{J/T}$	
proton magnetic moment	$\mu_p = 1.410\ 606\ 797\ 36(60) \times 10^{-26}\,\text{J/T}$	

Miscellaneous Physical Data

gravitational field strength (sea level)		$g = 9.81\,\text{N/kg}$
sun:	mass $= 1.99 \times 10^{30}\,\text{kg}$	radius $= 6.96 \times 10^{8}\,\text{m}$
earth:	mass $= 5.97 \times 10^{24}\,\text{kg}$	radius $= 6.37 \times 10^{6}\,\text{m}$
moon:	mass $= 7.35 \times 10^{22}\,\text{kg}$	radius $= 1.74 \times 10^{6}\,\text{m}$

mean earth-sun distance $= 1.50 \times 10^{11}\,\text{m}$

mean earth-moon distance $= 3.84 \times 10^{8}\,\text{m}$

Metric Prefixes

centi	$\text{c} = 10^{-2}$		
milli	$\text{m} = 10^{-3}$	kilo	$\text{k} = 10^{3}$
micro	$\mu = 10^{-6}$	mega	$\text{M} = 10^{6}$
nano	$\text{n} = 10^{-9}$	giga	$\text{G} = 10^{9}$
pico	$\text{p} = 10^{-12}$	tera	$\text{T} = 10^{12}$
femto	$\text{f} = 10^{-15}$	peta	$\text{P} = 10^{15}$

Contents

List of Tables

Additional Problems

A1. Repulsive and attractive tape. Take some magic Scotch tape and tear off a piece about 6 inches long. Attach it to the edge of a desk or table so that it hangs down. Then get another piece and stick the new (top) piece to the non-sticky side of the first (bottom) piece. Then quickly pull them apart and hang both of them from edge of the table. Label the strip that had been initially on the top "T" and label the other one "B."

(a) Next hold them so that the ends can come near each other. Do they attract? Repel? Neither? Try to write down a brief explanation.

(b) Take two more pieces of tape and repeat the above so that you have two T's and two B's. Which pairs attract and which repel? Why?

A2. Repulsive balloons. Blow up two balloons (not quite as full as possible) and tie them so they don't leak. Take a piece of string and attach (tie or tape) a balloon to each end, then hold the string in the middle and let the balloons at each end hang down.

(a) Now charge up the balloons by rubbing them against a fuzzy sweater or whatever you're wearing. Be sure to turn the balloon so that all parts get charged. Holding the string in the middle (and keeping your body parts away from the balloons!), what do you observe about the way the balloons hang? Make a sketch. Then make a force diagram for one of the balloons, showing all of the forces acting on the balloon (tension, electric and gravitational) and their relative magnitudes. **Note:** Since the balloons are in equilibrium, your force vectors should add up to zero. Make sure that they are drawn to reflect this.

(b) Now, carefully grab one balloon and push it slowly toward the other. Make sure that the balloons stay at the same height, so that a line through their centers remains roughly horizontal. What happens to the distance between the balloons as the string on the non-held balloon makes bigger and bigger angles with the vertical? When you have moved the held balloon enough so that the distance between the balloons is clearly different from that in part (a), make a force diagram for the non-held balloon, showing all the forces and their relative magnitudes. (The sizes of the arrows should also reflect any changes in the forces with respect to part (a).) **Comments:** Again, the balloon is in equilibrium, so the force vectors should add up to zero. But since your tension force is closer to horizontal now, what has to happen to its magnitude to keep the net force in the y-direction zero? What has to happen to the electric force to keep the net force in the x-direction zero? Make sure that your diagram is consistent with your observation about the distance between the balloons.

A3. Attractive balloons. The figure shows two balloons, one with a positive charge q and one with a negative charge q, each hung from the ceiling by a thread, as shown. The system is in equilibrium, and each thread makes an angle θ with respect to the vertical. Show that the charge q on the balloons is given by $q = x\sqrt{\frac{mg\tan\theta}{k}}$.

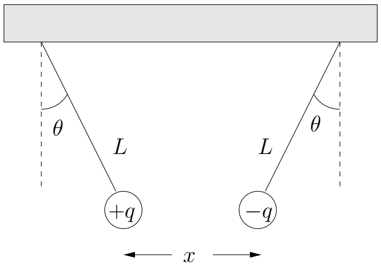


Figure 1: Problem A3

A4. Electric field simulations. Do Problem 1 on Electric Fields, accessible from the Lecture 1 calendar page.

A5. More E-Field Simulations. Do Problem 2 on Electric Fields, accessible from the Lecture 1 calendar page.

A6. Addition of electric fields. Two charges, each +5 μC, are on the y-axis, one at the origin and the other at $y = 6\,\text{m}$. Find the electric field on the y-axis at (a) $y = -3\,\text{m}$, (b) $y = +3\,\text{m}$, and (c) $y = +9\,\text{m}$.

A7. Determining $\vec{E}$ using integrals — semi-infinite rod. Point **P** is located a distance a from the bottom of a semi-infinite rod with uniform linear charge density λ.

(a) Calculate the $x-$component of the electric field at point **P**; you should set up and work through the integral.

(b) Calculate the $y-$component of the electric field at point **P**; you should set up and work through the integral. (Somewhat surprisingly the x and y-components have the same magnitude.)

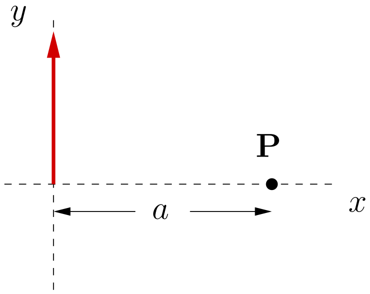


Figure 2: Problem A7

A8. Determining $\vec{E}$ using integrals — arc. The figure shows a quarter of a ring with radius R in the x-y plane with a total charge q, which is distributed uniformly over the quarter-ring. Working through the appropriate integral, determine the electric field at the origin (point **P**).

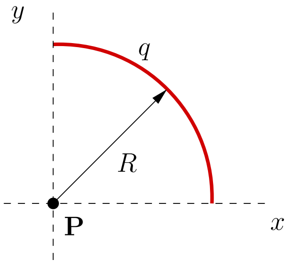


Figure 3: Problem A8

A9. Computer monitors. In a computer monitor (the old-fashioned tube-type monitor, not the flat panel ones), electrons are fired from the back toward the screen. Assume that the electrons start from rest and are accelerated through a potential difference of 50,000 V.

(a) What is the kinetic energy of the electrons when they hit the screen in electron volts?

(b) in Joules?

(c) What is the speed of impact of the electrons with the screen of the monitor? (You can neglect relativistic effects here, although you should note that the speeds aren't *that* much below the speed of light.)

A10. Directions of electric fields from continuous sources. The figure shows half of a ring in the x-y plane with a varying charge density λ which depends on the angle θ from the $+x$-axis.

(a) Determine the **direction** of the electric field at the origin (point **P**) if $\lambda = \lambda_0 \sin\theta$.

(b) Determine the **direction** of the electric field at the origin (point **P**) if $\lambda = \lambda_0 \cos\theta$.

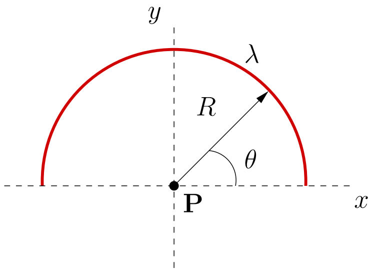


Figure 4: Problem A10

A11. Potential and electric field. The sketch shows three large parallel plate conductors held at the potentials shown. (Assume that the plates are infinitely large.) Find the direction and magnitude of the uniform electric field in each of the two interior regions.

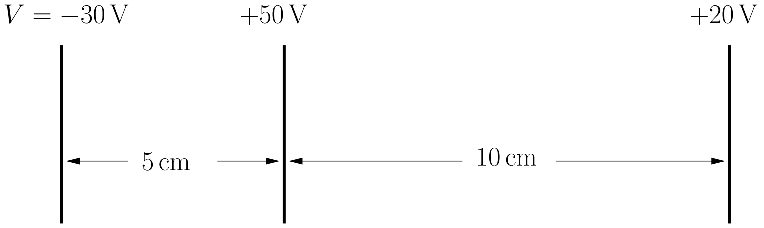


Figure 5: Problem A11

A12. Resistance and Dimmer Switches.

(a) Make a circuit with two batteries and a light bulb, all in series, and note the brightness. Now add a full length of one of your nichrome wires (bare silvery wire) to your circuit between the positive side of your batteries and the bulb. What happens to the brightness of the bulb? Explain briefly.

(b) Repeat with the other equal length piece of nichrome wire. Which wire has a larger resistance? Explain how you know. Examine the two wires carefully. Do you note any difference?

(c) Now use just the higher resistance wire in the circuit. Carefully unclip the lead on the wire nearest the battery, and touch it onto the wire at various places. How is the bulb's brightness affected? This is basically how a dimmer switch works!

A13. Flashlight circuit Make a circuit with two batteries and two bulbs, all in series.

(a) Now replace one of the bulb holders in the circuit with the higher resistance nichrome wire, and slide the alligator lead from the batteries along the wire (see Prob. A12) until the remaining bulb is about as bright as before with the two bulbs. This means the resistance of that length of nichrome is the same as a bulb. Measure this length.

(b) The wire has a diameter of 0.25 mm, and the resistivity of nichrome is $\rho = 1.0 \times 10^{-6}\ \Omega\cdot\mathrm{m}$. Calculate the resistance of the length of wire, and therefore of the bulb. **Continued→**

(c) From the battery voltage and the resistance that you just determined for one bulb, determine the current through the original circuit with two bulbs and two batteries.

A14. Fun with magnets. Take two cylindrical magnets and let them come together (N of one and S of other) around the end of a piece of thread (see Fig. 6). You'll end up with what is effectively one long magnet (from the two in combination) with a string coming up from in between the two (now connected) magnets. Hold the other end of the string and let the magnet dangle. Now gently twist the string from the top. Is there a preferred orientation for the magnet? What is this orientation? Explain briefly.

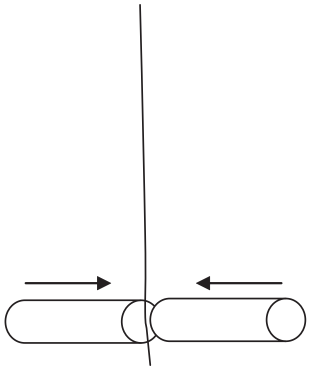


Figure 6: Problem A14

A15. Mapping the field of a permanent magnet.

(a) Draw a sketch of a cylindrical magnet including the direction of the magnetic field in the surrounding region. Mark the direction of the magnetic field with little arrows at various locations around the magnet.

(b) Now put your cylindrical magnet down on a flat surface and keep it from rolling. Next place your compass in various positions around the magnet and draw a sketch of what you observe. Your sketch should show a long rectangle (the magnet) surrounded by circles (the compass in various locations) with an arrow on each circle showing the direction that the red part of the compass needle pointed there. How did your measurements compare to your prediction?

A16. Destroying your TV or computer monitor. (Optional) BE CAREFUL with this one. A strong permanent magnet can magnetize parts of your monitor. This causes distortion of the images in both shape and color. Most monitors automatically can get rid of these extra fields using something called a "de-gausser", but the de-gausser doesn't usually activate except when the monitor is turned on (and is cold), so it might not fix the damage for a while, assuming the de-gausser is working. Thus the caution. Most monitors receive no permanent damage, but you never know, so don't go wrecking someone else's stuff.

Bring one of your magnets close to a CRT, like a computer monitor or TV. Move the magnet around the front and along the top and sides. Write down your observations. Why does this happen? What is a CRT anyway?

A17. Magnetic force on a moving particle. The figure shows a charged particle (with charge $-5.0\,\mu\text{C}$) moving with a speed $1.3 \times 10^5\,\text{m/s}$ in a vertically-directed magnetic field with magnitude $0.8\,\text{T}$. The particle is moving at an angle 30° with respect to the horizontal. Calculate the magnitude of the magnetic force on this particle, and state the **direction** of the force.

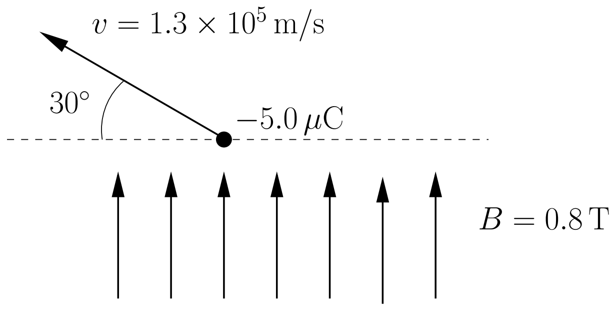


Figure 7: Problem A17

A18. Rail guns. A very simple application of magnetic forces can be used to make a nifty device called a “rail gun.” These have been proposed as devices that could be used to send back to earth (or the moon) raw materials mined from an asteroid. The idea is shown in the figure below. A bar of mass m can slide on two parallel rails separated by a distance H, and the whole apparatus is in a uniform magnetic field with magnitude B pointing into the plane of the paper. The rails are hooked up to a current source, and a current I passes through the rails and through the bar, which experiences a force that accelerates it down a track of length D. A bucket can be put on the bar, and anything in the bucket gets thrown off when the bar reaches the end of the rail.

Given the information above, determine the speed of the bar just before it reaches the end of the rails. (Hint: Think work.)

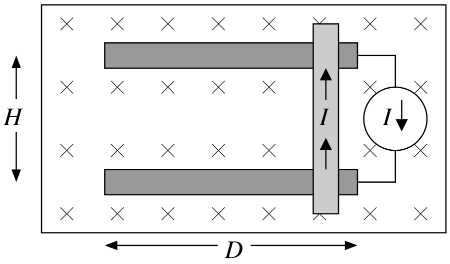


Figure 8: Problem A18

A19. Canceling the Earth’s magnetic field. A 100-turn circular coil with radius 30 cm is to produce a field at its center that will just cancel the earth’s magnetic field at the equator, which is 7.0×10^{-5} T directed north (horizontal). Find the current in the loop and make a sketch that shows the orientation of the loop and the current.

A20. Why aren’t there magnetic fields all around your house?

(a) The wire to a 100 Watt lamp carries about 1 amp of current to light the lamp. Calculate the magnetic field 5 mm from a wire carrying a steady current of 1 amp. Compare this to the Earth’s field of approximately 7×10^{-5} T.

(b) Put your compass near various wires in your room leading to electrical appliances and lights. Describe the compass deflections. Why don’t you see much? (Hint: The power cords aren’t just one wire but are a pair that carry the current both **to** and **back from** the appliance/light.)

A21. An electromagnet. Take your large and small nail and put them on a table top. Clip one end of an alligator lead to a battery (or two in series). Clip another to the opposite side of the battery. Wrap the piece of red insulated wire many, many times around the larger nail. Hold the wrapped nail in one hand and using the other, hook the leads from the batteries onto the ends of the wrapped wire to complete the circuit. You're making an electromagnet!

Try to pick up the small nail with the point of the large nail. Once it is off the table, disconnect the leads and see what happens. On your paper, explain briefly why the nail becomes a magnet.

A22. Solenoid. A solenoid with 2000 turns is 1.3 m long, has a radius 2.5 cm and carries a current of 3.0 A. What is the approximate magnitude of the magnetic field on the axis near the center of the solenoid?

A23. Electrical generation. Commercial electricity generation uses big strong magnets, multi-turn coils, and rapid relative motion. Suppose you could move a large 1.0 T magnet over the face of a 10 cm diameter 200-turn coil. What time interval between maximum flux and no flux would you need to produce an average emf of 120 V?

A24. Faraday with derivatives. A ring with radius r and resistance R lies in the x-y plane. The ring is located in a spatially uniform, but time-dependent magnetic field $\vec{B} = B_0(2-t+3t^2)\,\widehat{k}$. Calculate the magnitude of the electrical current $I(t)$ in the ring as a function of time.

A25. Eddy damping and brakes. Describe how eddy currents could be used to make brakes for a train or a car that don't require any friction.

A26. The Electromagnetic Spectrum. Consider the different types of electromagnetic waves: radio waves, microwaves, infrared, visible light, ultraviolet light, x-rays and gamma rays. Why do you think there are different names for these different kinds of electromagnetic waves? Is there a meaningful distinction? What are the differences between, say, gamma rays and infrared waves?

A27. Electromagnetic waves and you . Think about your life, your experiences and things which you may *not* have experienced but which clearly affect your life. There are numerous situations in which electromagnetic waves play a significant role in affecting your life. List **at least** four distinct and unique examples. These list might include things like examples of modern technology which rely on electromagnetic waves, as well as important electromagnetic wave phenomena which occur in the natural world.

A28. Falling loops. A long rectangular wire loop with mass m and resistance R is falling out of a region with a magnetic field with magnitude B_0 that is pointing out of the plane of the paper. The rectangle has a width w and a length that is so large that part of the loop is inside the region with the magnetic field and part is outside that region;

(a) Determine the current in the wire when the rectangle is falling from the magnetic field with a speed v.

(b) When dropped from rest the speed of the loop increases from 0 up to the terminal velocity, and then remains constant at this terminal velocity. Determine the terminal velocity for the falling loop. (Assume that the terminal velocity is reached before the top of the loop leaves the magnetic field.)

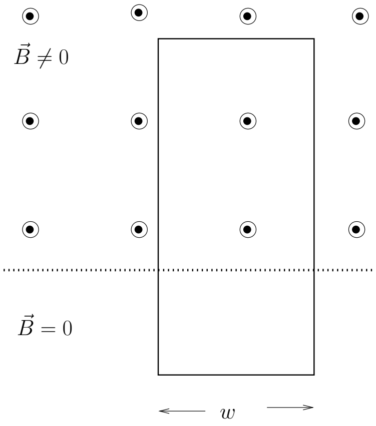


Figure 9: Problem A28

A29. Fetal heartbeat monitors. The *Babycom*$^{\text{TM}}$ *Home Doppler Fetal Heartbeat Monitor* uses ultrasonic waves with frequency 2.5 MHz, and can measure Doppler shifts as low as 100 Hz. Calculate the approximate speed of the fetal movements (blood flow) that account for these 100 Hz frequency shifts. (The speed of sound in air is 340 m/s, and the speed of sound in soft tissue — and water — is about 1500 m/s.)

A30. Electromagnetic waves. Consider an EM wave with electric field $\vec{E} = (6.6 \times 10^4\,\mathrm{N/C})\cos(1.2\pi x - 3.6 \times 10^8 \pi t)\,\hat{\jmath}$ with x in meters and t in seconds. Find the expression for the corresponding magnetic field.

A31. Re-radiated Light Waves. Consider a beam of light propagating along the y-axis. It is polarized along the z-axis as it enters a diffuse vapor cloud.

(a) Along which axis does the electric field of the light wave oscillate ($\pm x$, $\pm y$, or $\pm z$)? How do you know?

(b) Along which axis does the *magnetic* field of the light wave oscillate ($\pm x$, $\pm y$, or $\pm z$)? How do you know?

(c) The wave strikes some electrons in the vapor cloud, and they start oscillating. Along which axis do the electrons oscillate ($\pm x$, $\pm y$, or $\pm z$)? How do you know?

(d) These oscillating electrons generate a new re-radiated light wave. Along which axis does the electric field of the re-radiated wave oscillate ($\pm x$, $\pm y$, or $\pm z$)? How do you know?

(e) The re-radiated light wave cannot propagate in all directions. Along which axis (axes) can the new wave propagate ($\pm x$, $\pm y$, or $\pm z$)?

(f) For each answer to part (e), give the axis along which the *magnetic* field of the re-radiated wave oscillates ($\pm x$, $\pm y$, or $\pm z$).

A32. Traveling Waves on a Magic Spring. Take your Magic Spring and attach one end to a door knob, bed post or a friend. Hold the other end and walk away so the spring is stretched to about 6 feet. Grab a few of the coils and pull them toward you and release so that a longitudinal compression wave travels along the spring, bounces off the other end, and goes back and forth. See how far the pulse goes in five seconds (# of round trips times round-trip distance) and estimate the wave speed. Repeat by pulling the coils sideways to get a transverse wave, and find its wave speed. Are they different? Compare to the speeds found in Question A33.

A33. Standing Waves on a Magic Spring. Find a friend to help you time and count Magic Spring standing wave oscillations.

(a) Hold the spring with one end in each hand and generate standing waves as demonstrated in lecture. Get at least the fundamental and the next two higher harmonics. (Prize if you can demonstrate the $n = 5$ mode or higher!) Notice how the frequency you need to use changes for as the mode number increases.

(b) For quantitative analysis, set up the spring just as you had it when you measured pulse speeds in Question A32. Set up the lowest frequency standing wave by shaking one end of the spring and count ten full oscillations while your friend measures the time for these ten oscillations. Determine the period, frequency, and wavelength for this mode. Repeat your measurements for the $n = 2$ and $n = 3$ modes, making sure that you keep the length of the spring (the distance between your "shaking hand" and the fixed end) approximately the same for all of your measurements.

(c) What is the ratio of the frequency of the $n = 2$ mode to the fundamental frequency? What is the ratio of the frequency of $n = 3$ mode to the fundamental? Is this what you expect?

(d) Use the wavelength and corresponding frequency for each mode to determine a wave speed. Compare these to the speed of transverse waves measured in Question A32.

A34. A musical octave. Use your slide whistle for this one.

(a) Extend the slide all the way out. Blow softly on the whistle and note the pitch. Now push the slide in as you continue to blow. What do you hear? Explain *why* you hear the changes in the pitch that you hear when the slide is pushed inward.

(b) Extend the slide all the way out again. Again, blow softly on the whistle and note the pitch. Now continue to play the whistle as you push the slide inward, and stop when you reach a note that is an octave above what you started with. (This may take many tries to match.) What determines how far the slide should be pushed in to increase the pitch by one octave? Try to see if you can make a simple (approximate) measurement to verify your theory.

A35. Annoying your roommate. Here is another opportunity to play with your slide whistle.

(a) Play the slide whistle softly with the slide mostly or all the way out. Make a sketch of the standing wave in the air column associated with this note.

(b) Now blow harder on the slide whistle until the pitch changes. How is the frequency of this new (and most assuredly less-pleasant-sounding) note related to the one from part (a)? Draw a sketch of the standing wave to support your answer.

(c) Now plug your ears and blow even harder. You should be able to get the pitch to jump again (and to attract any stray dogs that are in the neighborhood, along with a few irate hall-mates). How is the frequency of *this* note related to the one from part (a)? Draw a sketch of the standing wave to support your answer.

A36. Organ pipes. Consider a 10 m organ pipe, open at both ends.

(a) Sketch the standing wave pattern for the three longest wavelength modes.

(b) From each sketch, determine the wavenumber k and the wavelength λ.

(c) Given your answers for (a) and (b), and given the speed of sound in air (340 m/s), determine the frequency of each mode.

A37. Stringed instruments. The lowest note that can be played on a bass fiddle is E1 (frequency 41.2 Hz) on a string of length 1.2 m (secured at both ends).

(a) Sketch the standing wave pattern for the three longest wavelength modes.

(b) From each sketch, determine the wavenumber k and the wavelength λ.

(c) Determine the wave speed for this string.

A38. Beats. Find someone else who is taking PHYS 212. One of you should blow (softly) your slide whistle with the slide somewhere in the middle. Hold the note while the other person blows his/her whistle and slowly adjusts his/her slide around the position of your slide. Listen for the beats and comment on how they change as the frequency of the second whistle is varied.

A39. Polarized light. Take your polarized disks and look at the following things and comment on what you observe:

(a) Go outside on a sunny day when the sun is low in the sky, preferably early or late in the day and look straight up at the sky, looking through one of the Polaroid disks. Then turn the disk. What do you observe? What does this tell you about light scattered from the sky?

(b) Look at various different LCDs (liquid-crystal displays): your calculator, a laptop or flat-screen monitor, a gas station pump readout, etc. Rotate the Polaroid disk and comment on what you observe. What does this tell you about liquid-crystal displays?

(c) How could you determine definitively if some sunglasses that you were buying at Wal-Mart^{TM} were polarized? (Don't trust the labels: my wife once bought "polarized" sunglasses that turned out to be fakes.)

A40. Soap films and bubbles.

(a) Here's one to try in the shower! Get your hands very wet and soapy. Then slowly slide your forefinger along your thumb stretching a soap film bigger and bigger. You can catch it on your other hand as demoed in lecture to make a big thin film. In fact, if your hands are wet, you can catch soap bubbles without popping them. (This is a great thing to do on a day just after it has rained — blow a bunch of soap bubbles on the wet pavement. They won't pop.)

If you are not good at this, try using bubbles from the soap bubble bottle in your kit. Blow a big bubble, catch it on the wand, and hold it while it thins out.

Now with the light behind you, look at the reflections off the film. You should see some really cool patterns of colors! Try to sketch the patterns (later) and show where each color is. In a few sentences, try to explain what you see in terms of thin films that you studied in class.

(b) (Optional if weather cooperates.) After it has rained, look at the pavement in the street or parking lots where cars are or have been. You might see a blotch of color. This usually indicates that a drop of oil has fallen on the ground. (A good place to look for these splotches is underneath cars driven by Bucknell faculty.) If you see one of these colorful oil spots, try stepping on it or brushing your foot over it. Comment on what you see and try to explain what you see in terms of the thin films that you studied in class.

A41. Fiber optics toy. Load up your fiber optics flashlight toy (Galaxy Wand) with the two little batteries (this may have already been done for you). Turn it on and notice how the cool colors of light come out the ends of the fibers but not out the sides. Take one of the fibers and bend it until it kinks. (If it breaks, try again.) Some of the light in that fiber will escape. Why does it escape? Why doesn't it escape if you make smooth curves in the fiber rather than a kink?

A42. Thin film interference. Assume that light with a wavelength λ_0 in air falls on a thin film composed mostly of water (index of refraction n_w) and with a thickness t. Assume that the water is on top of a mirror which inverts the EM wave when it reflects it.

(a) What is the wavelength of the light inside the film (in terms of λ_0 and n_w)?

(b) What is the path length difference between the light reflected off the front of the film and that which goes through the film, reflects off the mirror, and *then* comes back through the film and out the front?

(c) Are either (or both) of the two beams of reflected light inverted due to the reflection?

(d) What is the phase difference between the two reflected beams? (Hint: use the wavelength of the light *inside* the water film.

(e) If $n_w = 1.33$ and the film has a thickness $t = 300\,\text{nm}$, what is the largest wavelength of light (when in air) for completely destructive interference? For completely constructive interference?

A43. Son of thin film interference. Assume that light with a wavelength λ_0 falls on a thin film composed mostly of water (index of refraction n_w) and with a thickness t. Assume that the film is surrounded by air on both sides.

If $n_w = 1.33$ and the film has a thickness $t = 300\,\text{nm}$, what is the longest wavelength of light (when in air) for completely destructive interference in the reflected light? For completely constructive interference?

A44. Interference in music. Sitting at your desk, you are listening to music from your stereo. Your right ear is 2.2 m from one speaker and 2.6 m from the other. Nostalgic for your formative childhood years, you are listening to a *Barney's Greatest Hits* CD, and there is a particular solo by Barney in which the frequency of his tune ranges up to 1800 Hz. Ignoring reflections, determine the two smallest frequencies at which there will be destructive interference of the sound at your right ear; i.e., where the sound will be weakest.

A45. DVDs. If you shine laser light with wavelength 650 nm at normal incidence onto the surface of DVD, you'll find first-order intensity maxima coming back from the surface at an angle about 65° from the normal. Use this information to estimate (a) the spacing between adjacent grooves on a DVD (we know, it's one big spiral, but we want the spacing between the adjacent loops in the spiral); and (b) the approximate number of grooves (or loops of the spiral) on the disk, assuming a radius of approximately 4 cm for the disk.

A46. Compact disks. Take a copy of your favorite CD (or your least favorite one – it doesn't matter) and look at the groovy side in strong light. Tip the CD at different angles. Why do you see colors? Do they change for different tip angles? Explain briefly.

A47. Resolution. Go out to the Academic Quad at the end opposite the library. Looking at the clock on the library tower, walk toward the library until you can just distinguish the five separate lines that make up the Roman numeral 8 (VIII). Now pace off your distance to the library doors. Add 30 ft to get the distance between your former position and the clock face. (Assume that each pace is about 75 cm.)

Now using Rayleigh's criterion, calculate the minimum separation of objects on the clock face that you could distinguish with the naked eye. Assume that the visible light you use has a wavelength of 500 nm. How does your calculated separation compare with Prof. Bowen's estimate of the actual separation between lines (5 cm)?

A48. Electromagnetic traveling wave. The electric field of a traveling wave is represented by

$$\vec{E}(x,t) = 6 \times 10^4 \cos(0.5 \times 10^7 x - 1.5 \times 10^{15} t)\,\hat{k}$$

with E in N/C, x in meters, and t in seconds.

(a) Calculate the wavelength, period, and wave speed of this wave. What kind of EM wave is this?

(b) Determine the amplitude of the associated magnetic field wave (in Tesla).

(c) Determine the direction of propagation for this wave.

A49. Wave nature of light. Here's an easy way to see a manifestation of the wave nature of visible light. Put on your diffraction glasses and look at a small bright light source. Sketch the pattern in your notes. Explain briefly why this pattern is indicative of the light's wave properties.

A53. Implications of photoelectric effect. Professor Quack doesn't believe in quantum theory. To make his point he does a photoelectric experiment. He takes a white light source and two filters — one filter allows blue light to pass, and one filter allows green light to pass. He measures the current when the blue filter is in place and finds that it is *less* than the current he measures with the green filter in place. He argues that according to quantum theory blue photons should have more energy than green photons, so the blue light should result in a greater current than that caused by the green light. What do you have to say to Professor Quack?

A58. Proton in a box. A proton is confined to a one-dimensional infinite potential well with a width of 2×10^{-14} m. Assume that it is in its first excited state (i.e., not the ground state, but the next state).

(a) Draw a sketch of the wavefunction ψ corresponding to this state, and indicate the locations where you would never expect to find the proton.

(b) Determine the wavelength of the quantum wave associated with the proton.

(c) Using your answer to part (b), determine the magnitudes of the momentum and energy of the proton in the first excited state.

A59. Electron in a box. An electron is confined to a one-dimensional infinite potential well of width b, and is in its second excited state. The energy of the electron is 75 eV. (Remember that the energy is all kinetic in this case.)

(a) Calculate the momentum of the electron.

(b) Use your result from part (a) to calculate the wavelength of the electron.

(c) Draw a sketch of the wavefunction ψ corresponding to this state, and indicate the locations where you would never expect to find the electron.

(d) Use your answers from parts (b) and (c) to calculate the width of the potential well b.

A60. Whirl-a-Tune and quantization. Take your “whirl-a-tune” tube and hold the end with the little neck. Then whirl the tube slowly over your head and listen for a tone. Whirl faster and slower and note the pitch that you hear. Can you get any frequency or just certain discrete tones? Explain briefly why only certain frequencies are heard. The “quantization” of frequencies that you hear in this case involves the same sort of wave mechanics as the quantization of energy levels in a (small) confined system.

A61. Uncertainty. A wide beam of particles with mass m and horizontal momentum $\vec{p}_1$ is sent toward a vertical slit of width a. Consider one particle that you know makes it through the slit, but you know absolutely nothing else about the particle.

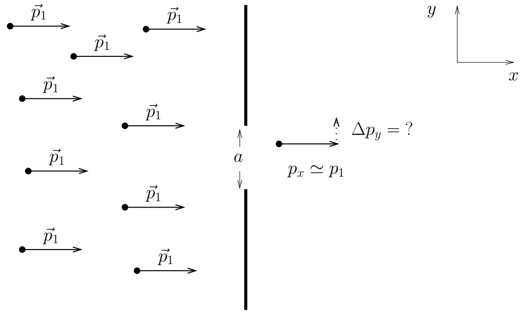


Figure 10: Problem A61

(a) What is the approximate uncertainty in the vertical position of the particle right after it passes through the slit? (Don't make this harder than it is; the answer should be obvious.)

(b) Determine the approximate minimum uncertainty in the vertical momentum of the particle right after it passes through the slit.

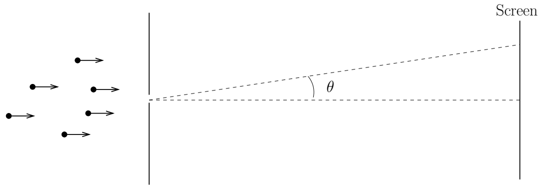


Figure 11: Problem A61

(c) If the particle is detected on a faraway screen, show that you expect to find particles hit the screen over a range of angles $\pm\theta$, where $\theta \simeq \tan^{-1}[\lambda/(2\pi a)]$.

A62. More uncertainty. The position of a macroscopic particle of mass 0.04 kg is measured to an accuracy of $\approx 10^{-12}$ m. (This can actually be done using interferometers.) Compute the minimum uncertainty in the velocity of the object.

A63. Magic Springs and the Particle-in-a-Box. Hold both ends of your Magic Spring, and get standing waves in the first, second, and third lowest frequency modes. Sketch the wave patterns and compare them to the wave functions for the three lowest energy states of a "particle in a box."

A64. Schrödinger equation for a classically allowed situation. Consider a particle of mass m in a region in which the potential energy is constant, i.e., $U(x) = U_0$, and assume that the total energy of the particle E is greater than the potential energy, i.e., $E > U_0$. (This is the case for classically allowed motion.) To determine the wave function we must find a function $\psi(x)$ that satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

In this problem you will try three "guesses" for $\psi(x)$ and see if they satisfy the Schrödinger equation. The three "guesses" are

- $\psi_1(x) = Ax^2$
- $\psi_2(x) = B\sin(kx)$
- $\psi_3(x) = Ce^{-\kappa x}$,

where A, B, C, k, and κ are undetermined real constants.

(a) Rearrange the Schrödinger equation so that the second derivative $d^2\psi/dx^2$ is alone on the left.

(b) Plug $\psi_1(x)$ into the Schrödinger equation and see if there is any choice for the constant A that will make $\psi_1(x)$ satisfy the equation for all values of x.

(c) Plug $\psi_2(x)$ into the Schrödinger equation and see if there is any choice for the constants B and k that will make $\psi_2(x)$ satisfy the equation for all values of x .

(d) Plug $\psi_3(x)$ into the Schrödinger equation and see if there is any choice for the constants C and κ that will make $\psi_3(x)$ satisfy the equation for all values of x.

(e) You should have found that $\psi_2(x)$ can be a solution for the proper choice of k. Determine the wavelength of the oscillations in terms of $\hbar$, m, E, and U_0. (i.e., solve for k and remember from the waves unit that $k = 2\pi/\lambda$.) Is your result consistent with that predicted from the de Broglie relationship? (Hint: $E - U_0$ is the kinetic energy $K = p^2/2m$. Re-write things in terms of the momentum and the answer should drop into your lap.)

A65. Schrödinger equation for classically forbidden situation. Consider a particle of mass m in a region with a constant potential energy U_0, and assume that the total energy of the particle E is *less* than the potential energy, i.e., $E < U_0$. (This isn't possible for classical motion, but continue anyway.) To determine the wave function we must find a function $\psi(x)$ that satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

In this problem you will try three "guesses" for $\psi(x)$ and see if they satisfy the Schrödinger equation. The three "guesses" are

- $\psi_1(x) = Ax^2$
- $\psi_2(x) = B\sin(kx)$
- and $\psi_3(x) = Ce^{-\kappa x}$,

where A, B, C, k, and κ are undetermined real constants.

(a) Rearrange the Schrödinger equation so that the second derivative $d^2\psi/dx^2$ is alone on the left.

(b) Plug $\psi_1(x)$ into the Schrödinger equation and see if there is any choice for the constant A that will make $\psi_1(x)$ satisfy the equation for all values of x.

(c) Plug $\psi_2(x)$ into the Schrödinger equation and see if there is any choice for the constants B and k that will make $\psi_2(x)$ satisfy the equation for all values of x.

(d) Plug $\psi_3(x)$ into the Schrödinger equation and see if there is any choice for the constants C and κ that will make $\psi_3(x)$ satisfy the equation for all values of x.

A66. Tunneling and time-energy uncertainty. Consider an electron hitting a barrier. Assume the electron has an energy $E = 50\,\text{eV}$, and the barrier has a height $U = 100\,\text{eV}$. Semi-classically, to tunnel through the barrier, the electron must "borrow" enough energy to get over the barrier, and must hold this energy long enough to travel the width of the barrier. The best-case scenario is if the energy fluctuates up to a value of $2U - E$ or $150\,\text{eV}$. (See "Optional problem" below if you want to see where this comes from.)

(a) Using the energy-time uncertainty relation $\Delta E \Delta t \approx \hbar$, approximate the typical duration of the energy fluctuation (i.e., determine Δt).

(b) Determine the classical velocity of the electron in the barrier region if it has an energy of $150\,\text{eV}$. (Warning: the kinetic energy of the electron is not $150\,\text{eV}$ here.)

(c) Determine the maximum width L of the barrier such that the electron will make it through in a time Δt.

(d) The width L that you have just determined is a width for which you would expect a reasonable probability for an electron to tunnel through a barrier. You can calculate the transmission probability T explicitly by $T = e^{-2\alpha L}$, where $\alpha = \sqrt{2m(U-E)/\hbar^2}$. Use your value of L and the information given above to verify that you get a reasonable value for T. (by "reasonable," we mean that you should get a probability greater than 0.1, but, of course, it must be less than 1.)

A67. Optional for mathophiles . . . (You don't have to hand this in.) For the preceding "A" problem, show that the semi-classical approach discussed for tunneling gives the largest value of the barrier width L if the particle borrows enough energy ΔE to get to an energy of $(2U - E)$. Hint: Use the approach from the previous problem to find the maximum barrier width L if the particle fluctuates up to an energy of E_{high}. Then take the derivative dL/dE_{high} and set this equal to zero to figure out the optimal E_{high}. Note: conceptually, the optimal energy is a small enough energy such that Δt is relatively long, but large enough such that the electron still has some kinetic energy while traveling across the barrier.

A68. Semi-infinite square-well potential. Download the Excel worksheet `semi-finite.xls` from either the *Handouts* page or from the Calendar page. This sheet shows the calculations for determining the wavefunctions for a potential well that is infinite at $x = 0$ but of finite magnitude on the right side of the well (which is at $x = 5$ in this problem). You'll see two graphs: the top one shows the semi-infinite potential well (in purple) along with a non-normalized plot of the calculated wavefunction so you can see it along with the potential. The bottom graph shows the normalized wavefunction, corresponding to the second-to-last column in the worksheet.

When you bring up the worksheet, the energy will be set for the value for the ground state. Some questions:

(a) Sketch or print out (just the first page!) the wavefunctions that are displayed for the ground state along with at least two of the excited states. To display the 1st and 2nd excited states, type in 0.64282 and 1.4144 respectively in the framed box for energy.

(b) What happens if you type in an energy that **isn't** one of the well-defined energies for the problem? Try it out, and comment on what happens. Had we not told you what the allowed energies were, how might you figure them out? (You'll be doing this in lab later this semester.) **Continued →**

(c) For any of the allowed states, show that the state plotted in the bottom graph is normalized. Hint: we have already created a column for the square of the normalized wavefunction (the right-most column). You might want to take advantage of the `sum(start:end)` routine in Excel.

A69. Classically allowed and classically forbidden probabilities. Load the `semi-finite.xls` worksheet (the same one from the previous problem). The ground state should be displayed initially.

(a) If a measurement were done on this system, what would be the probability that the particle would be found in the region $x < 5$? Explain briefly how you calculated this from the Excel worksheet. **Hint:** Remember that for a continuous wavefunction $\psi(x)$, the probability of finding the particle in a particular region is $\int_{x_1}^{x_2} P(x)\,dx = \int |\psi(x)|^2\,dx$. However, we don't actually have a wave function to integrate; we have a numerical solution instead. But we can do a *Riemann Sum* and add up the contributions: $\sum_i P(x_i)\,\Delta x = \sum |\psi(x_i)|^2\,\Delta x$. You'll have to determine what Δx is in this Excel worksheet.

(b) What would be the probability that the particle would be found in the region $x > 5$?

(c) What would be the answer to questions (a) and (b) if this was a classical particle in a semi-infinite square-well potential with energy less than U_0 (i.e., no quantum effects)?

(d) Answer questions (a) and (b) again, but for the second excited state (with $E = 1.4144$). Compare your answers with those for the ground state. Do the results make sense, considering the higher energy? Explain.

(e) What do you think would happen if the potential dropped back to 0 at $x = 6$? Would the particle remain trapped indefinitely? Explain why or why not, and refer to the graphs to support your answer. (You might want to sketch them or print them out.)

A70. Wavefunctions and potential energy. The illustrated graph gives the wavefunction for bound state of an electron in some one-dimensional potential well.

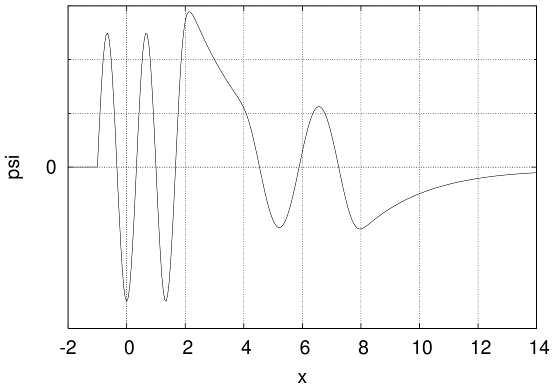


Figure 12: Problem A70

Make a qualitative sketch of the potential energy $U(x)$ versus x that could give rise to this wavefunction. Include an indication of the total energy E on your sketch.

A71. Wavefunctions and probabilities. Using the sketch below, of the wavefunction $\psi(x)$, identify which letters indicate locations where the particle is: (a) most likely to be found and (b) least likely to be found.

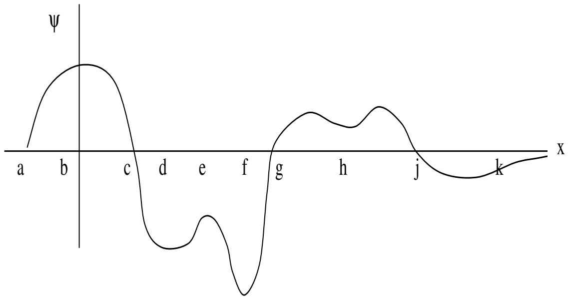


Figure 13: Problem A71

A72. Barrier tunneling: Calculating approximate probabilities. A 15 eV electron is incident on a potential barrier of height 22 eV and width of 0.05 nm.

(a) Use the transmission probability discussed in part (d) of Problem A66 to estimate the order of magnitude of the probability that the electron will tunnel through the barrier.

(b) If one million electrons with energy 15 eV hit this barrier, roughly how many of them would you expect to get through?

(c) Repeat parts (a) and (b) for a barrier width of 0.5 nm.

A73. Annoying your roommate, Part 2 or **Superposition of states.** Take your slide whistle and with the slide most or all the way out, blow gently into the whistle. As we discussed in the previous unit, the note that you hear is due to the fundamental mode of the slide whistle. If you blow harder, the pitch will jump to a higher value, corresponding to the second standing-wave mode.

It is possible to blow hard enough – but not too hard – such that you hear two notes at the same time. Do this, and comment on what you hear. Now, consider the analogous quantum problem. If these were two matter waves instead of sound waves, what would the different pitches that you hear correspond to?

A74. Life in a quantum world. Last semester we asked you to describe some relativistic effects that you would notice if the physical "speed limit" of light were 4 mph instead of 3×10^8 m/s. Now imagine that the quantum constant $\hbar$ were 1 Joule-sec.

(a) The uncertainty principle in this world would be $\sigma_x \sigma_p \geq \frac{1}{2}\,\mathrm{kg{\cdot}m^2/s}$. Imagine that someone throws a ball to you. Describe what you would experience trying to catch the ball.

(b) Consider the energy of a photon. With light frequencies still in the 10^{14} Hz range, describe how it would feel to be sunbathing on a beach.

(c) What would your approximate de Broglie wavelength be when walking? What would happen when you walk through a doorway?

A75. Transitions. The sketches below show the state of a two-level atom and possibly a photon. For each "Before" sketch make a corresponding "After" sketch and name the process.

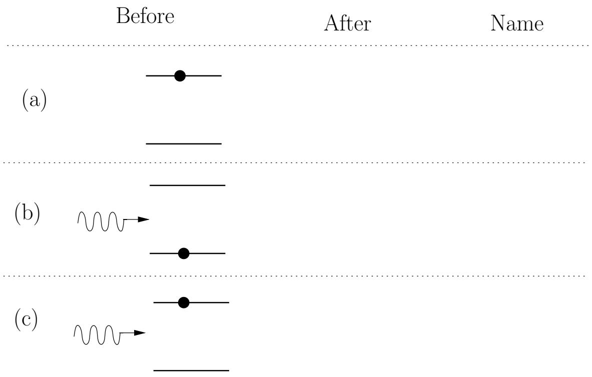


Figure 14: Problem A75

A76. Population inversion. Explain briefly why a population inversion is necessary for the operation of a laser.

A77. Superconductors. (Do in Problem Session) Here we investigate some magnetic properties of superconductors.

(a) Closely observe the little cube hovering over the disk. Comment on what you observe. What evidence do you have that this is a superconductor? Can you make the cube spin?

(b) Explain how the superconductor can levitate the magnet.

A78. Flipping magnets. Find a friend to help you explore magnetic resonance. Take your magnet and tie string around it so it is supported in the center and hangs horizontally when you hold the string above and below the magnet. Have your friend bring his/her magnet nearby and note that your magnet tries to align. Keeping the string fairly tight, use your hand to twist the magnet slightly. Are you putting energy into the system? What happens when you release the magnet? On the atomic scale, where does this released energy go?

A79. Electron spin resonance. What is the wavelength of a photon that will induce a transition of an electron spin from parallel to anti-parallel orientation in a magnetic field of magnitude 0.20 T? (From Halliday, Resnick, and Walker p. 1048.)

A80. Nuclear magnetic resonance. Electromagnetic waves with frequency $f = 34\,\mathrm{MHz}$ illuminate a sample that contains hydrogen atoms. Resonance is observed when the strength of the constant external magnetic field equals 0.78 T. Calculate the strength of the local magnetic field at the site of the protons that are undergoing spin flips, assuming the external and local fields are parallel there. (adapted from Halliday, Resnick, and Walker p. 1048.)

A81. Flipping inside atoms. The proton, like the electron, has a spin quantum number s of 1/2. In the hydrogen atom in its ground state ($n = 1$ and $l = 0$), there are two energy levels, depending on whether the electron and proton spins are parallel or anti-parallel. If the electron of an atom has a spin flip from the state of higher energy to that of lower energy, a photon of wavelength 21 cm is emitted. Radio astronomers observe this 21 cm radiation coming from deep space. What is the effective magnetic field (due to the magnetic dipole moment of the proton) experienced by the electron emitting this radiation? (From Halliday, Resnick, and Walker p. 1048.)

A82. MRI. Assume that the magnetic field along a line passing through a patient's brain in an MRI scan is described by the function $B(x) = 0.5 + 0.6x$, where B is in Tesla and x is in meters.

(a) What is the location in the brain where protons will flip in response to a 30 MHz oscillating magnetic field? (Give your answer as $x =$ ___ m.)

(b) If you want to probe a possible tumor at a position of $x = 0.50\,\mathrm{m}$, at what frequency should you oscillate the magnetic field?

(c) For your answer in part (b), what is the energy of the photons that are probing your patient (in eV)? Considering that the weakest molecular bonding energies are around 0.1 eV, is this safe for your patient?

A83. Particle decay. This exercise simulates the conversion of rest and kinetic energies in a particle decay. Take 10 coins (or any 10 objects – pencils, bottle caps, small elephants, ...) and lump them together on a table or desktop. Each item represents 1 unit of energy.

(a) Assume your pile represents a single massive particle with $m = 10$ in some units. Now assume this particle decays in to 2 particles with mass 5 and 4 units. Split your pile up into these two particles. How many extra items are left? What do these extra items represent?

(b) Start again with a single pile representing a single particle of rest energy 10 units. Now have the particle decay into two particles, one of rest energy 5 units, the other of 6 units. (No borrowing from friends!) Why can't you have a decay result in a larger total rest energy?

A84. Exchanging virtual particles. (OPTIONAL): Find a friend and a pen. The pen represents a virtual force carrier (messenger particle) that will be exchanged between you and your friend.

(a) Hold a pen in one hand, aiming the point of the pen toward your friend. The direction the pen points is the direction of the momentum of the messenger. Now give the pen to your friend and conserve momentum. To do this, you should each modify your motion to reflect the momentum exchange. For example, if you give away leftward momentum, you must move rightward to compensate. Describe your relative motions after the exchange of pen.

(b) Repeat, but have the pen initially pointing away from your friend.

A85. The expanding universe. Take a balloon and draw some stars, planets, and galaxies on the surface of the deflated balloon.

(a) Now blow up the balloon and watch how the distance between adjacent galaxies changes as the universe expands. Record your observations.

(b) With the balloon partially inflated, choose a reference galaxy. Find two objects nearby, with one about as twice as far from the reference galaxy as the other. Measure the distance. Then blow up the balloon and measure the distances again. How do their rates of change of distance compare? Compare this to Hubble's Law.

A86. A moving wave. A wave is described by $\psi(z,t) = 5\cos(\pi z/2 + \pi t/4)$, where z is in meters, and t is in seconds.

(a) Plot ψ versus z at time $t = 0$ between $z = -3$ and $z = 3$. Make another plot of ψ versus z at time $t = 1\,\mathrm{s}$.

(b) Find a point of zero displacement at $t = 0$. Where is this point of zero displacement at time $t = 1\,\mathrm{s}$? (Take into consideration the direction the wave is traveling.)

(c) Use your answers to parts (a) and (b) to calculate the speed of the wave. Does your answer agree with the speed determined from ω/k?

A87. Two antennas. Two antennas are $\lambda/4$ apart. Each emits a wave with the same amplitude and the same phase. A receiver is located far away from the antennas, but is placed such that the receiver and the antennas fall on a single straight line. Individually, each antenna gives a wave of amplitude A at the receiver. Calculate, in terms of A, the total amplitude at the receiver when both antennas are emitting.

A88. Find the third maximum. Laser light of wavelength 633 nm shines on a double slit arrangement with a slit separation of 0.003 mm. The interference pattern is viewed on a screen several meters away. At what angle θ does one observe the third maximum away from the central maximum?

A89. Two loudspeakers. Two loudspeakers, 3.0 m apart, are driven at the same frequency and in phase. They emit sound with a wavelength of 2.0 m.

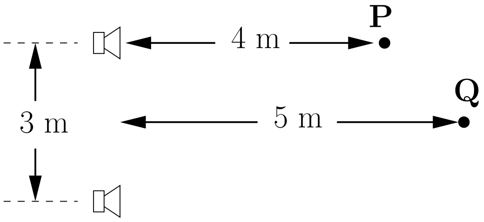


Figure 15: Problem A89

(a) Point **P** is 4.0 m from the line joining the speakers and is directly in front of one speaker. Is the intensity at **P** a maximum, a minimum, or neither?

(b) Point **Q** is 5.0 m from the midpoint between the speakers and equidistant from them. The intensity at **Q** when only one speaker is on is I_0, and when only the other speaker is on the intensity at **Q** is $4I_0$. Find the intensity at **Q** when both speakers are on.

A90. Three loudspeakers. Three loudspeakers are arranged on a line and separated by 3.0 m, as shown in Fig. 16. They are driven at the same frequency and in phase. They emit sound with a wavelength of 2.0 m. Point **P** is 4.0 m from the line joining the speakers and is directly in front of the central speaker. The sound intensity at **P** when any one speaker is on is I_0.

(a) Draw a phasor diagram representing the sound waves at point **P**. Include a phasor for the wave from each speakers, and a phasor for the total wave that results from the superposition of these three waves.

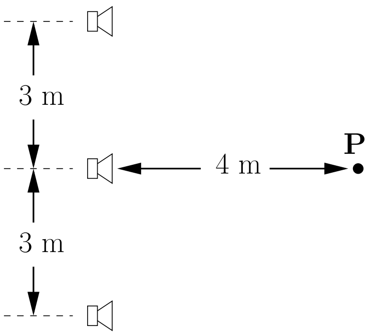


Figure 16: Problem A90

(b) Calculate the intensity (in terms of I_0) at **P** when all three speakers are on.

A91. Find the formula. The illustration shows two snapshots of a traveling wave, one taken at $t = 0\,\text{s}$ and one taken at $t = 0.25\,\text{s}$. Determine a formula for the function $y(x,t)$ that describes this traveling wave.

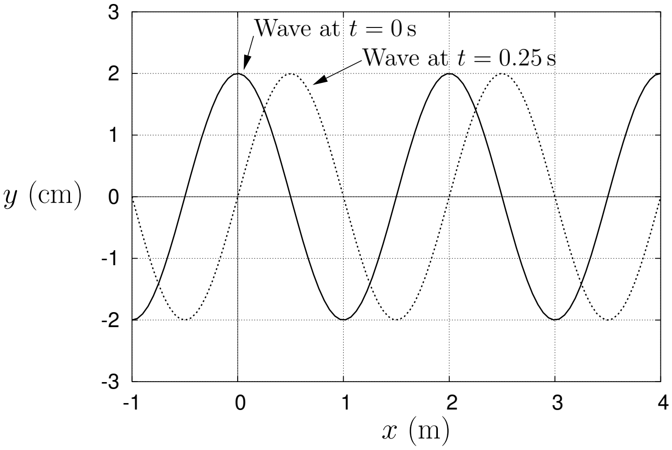


Figure 17: Problem A91

A92. Adding waves graphically. The graph in Fig. 18 shows a snapshot of two traveling waves at the same instant of time. The waves have the same speed and frequency. Add these two waves graphically to find their sum.

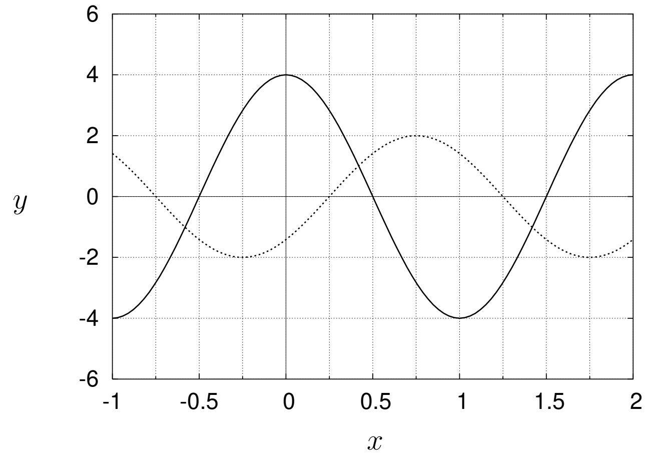


Figure 18: Problem A92

NOTE: There is no calculation to be performed in this graphical addition. You should be thinking about this as filling in entries in a table like the following by reading values from the graph, and then plotting the last column, y_{sum} vs. x on the graph.

x	y_{solid}	y_{dotted}	y_{sum}
-0.5			
-0.25			
0.0			
0.25			
etc.			

A93. Adding waves with phasors. The graph in Fig. 19 shows a snapshot of two traveling waves at the same instant of time. The waves have the same speed and frequency.

(a) Draw two phasors representing these two waves.

(b) Calculate the amplitude of the superposition of these waves.

(c) Calculate the phase shift of the resultant wave with respect to solid wave in the illustration.

(d) Compare your resultant amplitude and phase with the graphical result you got in Problem A92.

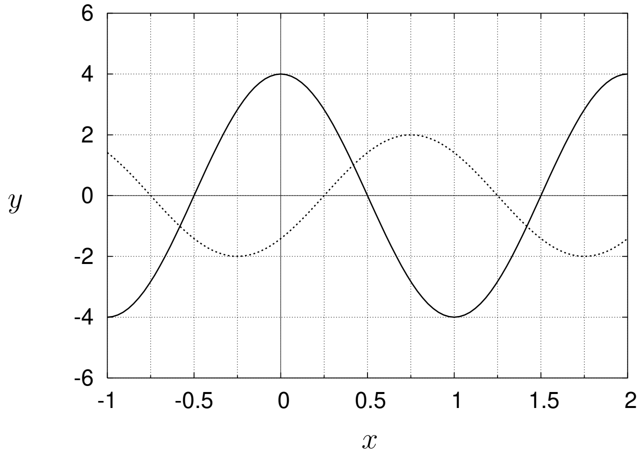


Figure 19: Problem A93

A94. Parking Lot. John, the aspiring physics student/parking attendant (see Supp. Ch. 3, Problem # 2) gets a job at a new hotel that has a more conventional parking lot. The parking lot has a rectangular shape on an x-y coordinate system with dimensions 100 m × 50 m, and the lot is divided into three sections, **A**, **B**, and **C** (see figure). Although the lot is more conventional, John still tells car owners the whereabouts of their cars in terms of probabilities and probability densities, only now the probability densities are given in terms of *probability per unit area* instead of of *probability per unit length*, and they are functions of two variables, x and y.

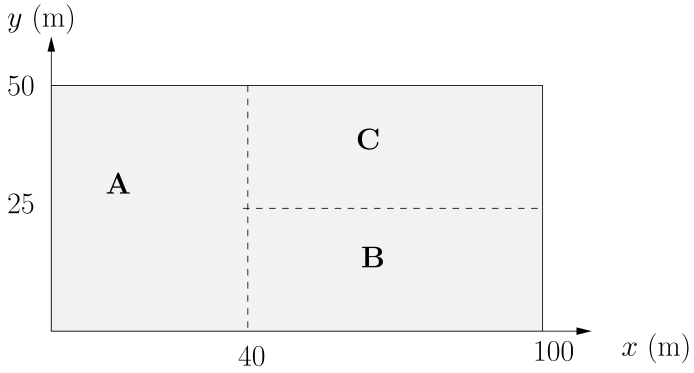


Figure 20: Problem A94

(a) Mr. Vanderbilt is told that his car "could be anywhere in the lot," which means that the probability density is constant everywhere (i.e., there is no difference between the sections). Calculate the value of this uniform probability density $P(x, y)$ for Mr. Vanderbilt to find his car at a position (x, y) on the coordinate system of the lot. (Your answer should be in units of probability/m^2.)

(b) Find the probability that Mr. V's car is in section **A** of the lot.

(c) Mrs. Reeve is told that the probability density to find her car is a constant P_A in section **A**, a second constant $P_B = 4P_A/3$ in section **B**, and a third constant $P_C = 2P_A/3$ in section **C**. Find the constants P_A, P_B, and P_C.

(d) Based on your results from part (c), calculate the probability that Mrs. Reeve's car is located in the lower left quarter of the lot, i.e, in the region where $0 \le x \le 50$ and $0 \le y \le 25$.

A95. Daughter of parking lot. John, the aspiring physics student/parking attendant (see Problems Supp. Ch. 3 #2 and **A94**) gets a job at this one has a circular parking lot with radius 40 m laid out on a r-θ polar coordinate system, and the lot is divided into two sections, **A** and **B**.

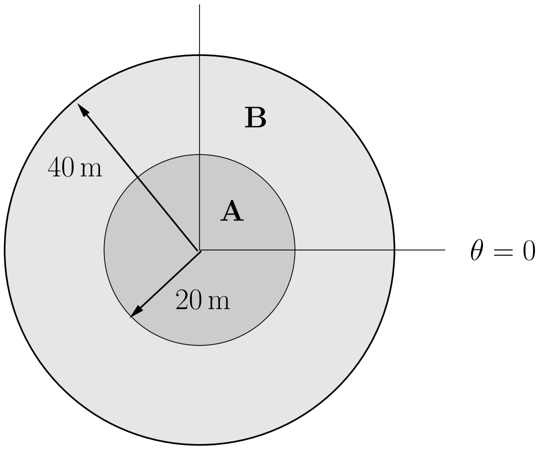


Figure 21: Problem A95

(a) Mr. Vanderbilt is told that his car "could be anywhere in the lot," which means that the probability density is constant everywhere (i.e., there is no difference between the sections). Calculate the value of this uniform probability density $P(r, \theta)$ for Mr. Vanderbilt to find his car at a position (r, θ) on the coordinate system of the lot. (Your answer should be in units of probability/m^2.)

(b) Find the probability that Mr. V's car is in section **A** of the lot.

(c) Mrs. Reeve is told that the probability density to find her car is a constant P_A in section **A** and a second constant $P_B = P_A/2$ in section **B**. Find the constants P_A and P_B.

(d) Based on your results from part (c), calculate the probability that Mrs. Reeve's car is located within 30 m of the center of the lot.

A102. Electric field from a ring of charge. A ring with a radius R and total charge Q (distributed uniformly) lies in the x-y plane. Determine the electric field at the point P on the z-axis at a height h above the center of the ring. **Show all the steps needed to set up and evaluate the integral.**

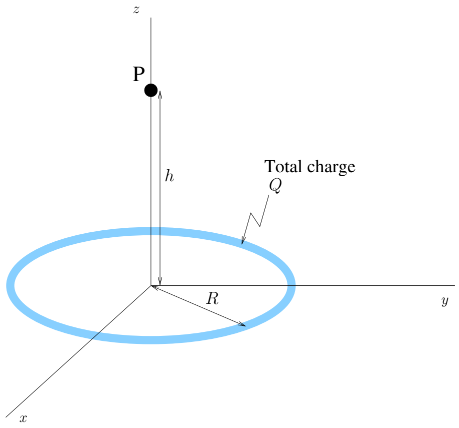
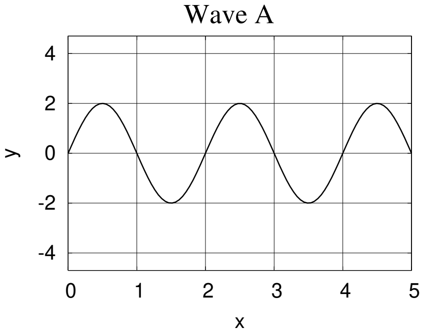
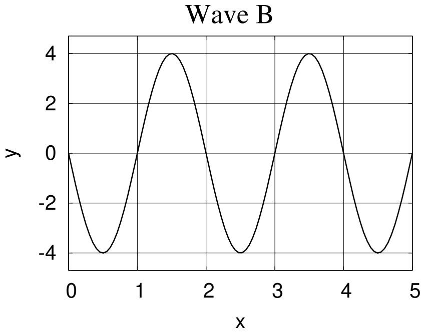


Figure 22: Figure for problem A102.

A103. Phasors and waveforms. The graphs below show snapshots of two traveling waves on a string at the same instant of time.

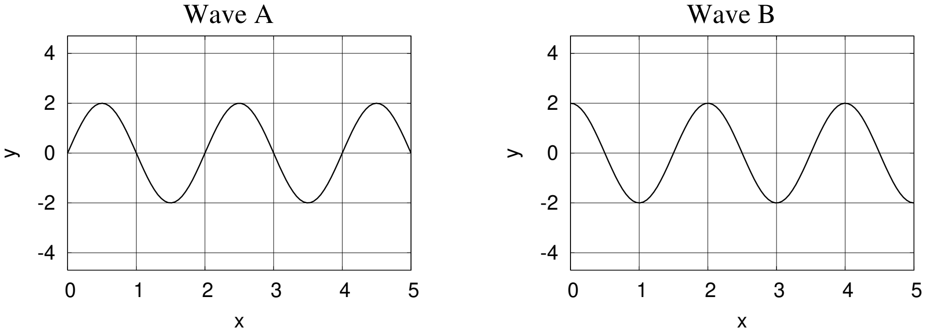




(a) Draw a phasor diagram which represents the superposition of these two waves. The diagram should be clearly labeled to show what represents “Wave A,” what represents “Wave B,” and what represents the superposition of the two (“A+B”).

(b) From the phasor diagram, determine the amplitude of the superposition of waves A and B.

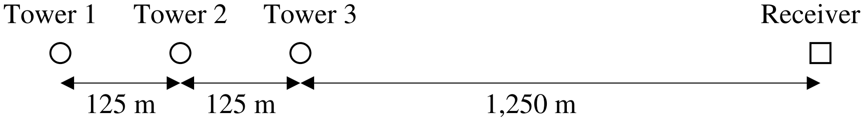
A104. Phasors and waveforms II. The graphs below show snapshots of two traveling waves on a string at the same instant of time.



(a) Draw a phasor diagram which represents the superposition of these two waves. The diagram should be clearly labeled to show what represents “Wave A,” what represents “Wave B,” and what represents the superposition of the two (“A+B”).

(b) From the phasor diagram, determine the amplitude of the superposition of waves A and B.

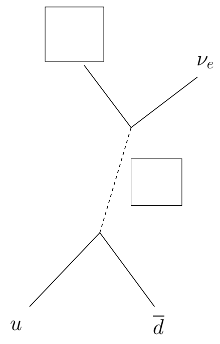
A105. Radio towers. Three equally spaced AM radio towers are located to the left of a receiver as illustrated. The towers broadcast in-phase radio waves of equal amplitude and with wavelength of 500 m.

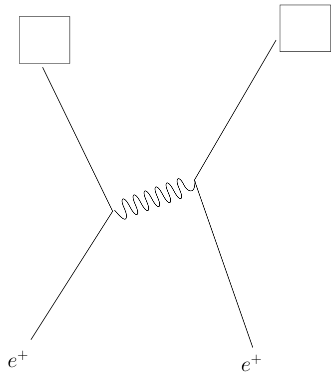


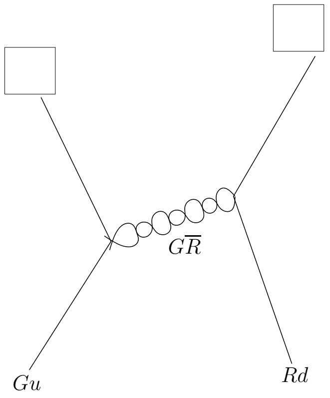
(a) Draw a phasor diagram representing the combined radio wave at the receiver.

(b) Assuming the amplitude of the wave reaching the receiver from each tower individually is A, determine the amplitude of the combined wave at the receiver.

A106. Feynman fun. Fill in the missing particles (including color and/or charge labels where necessary) for the following three Feynman diagrams.







A109. A square of charges. Three identical charges q and a fourth charge $-q$ form a square with sides of length a. Find the electric force vector acting on a charge Q placed at the center of the square.

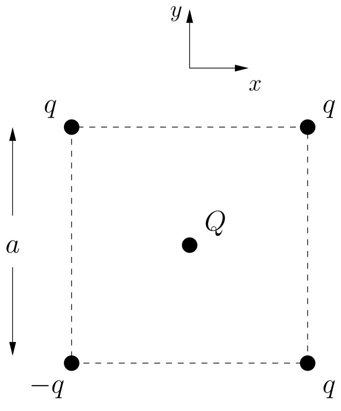


Figure 23: Problem A109.

A110. Parallel currents. Two parallel wires oriented perpendicular to the page are separated by 10.0 cm. Each wire carries a current of 1.5 A directed out of the page. Determine the magnitude of the total magnetic field at a point P in the figure, a distance 12.0 cm above the midpoint of the line connecting the two wires.

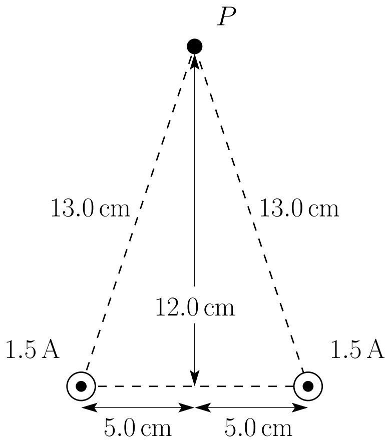


Figure 24: Problem A110.

A111. Magnetic field from a wire. A long straight wire 1.0 cm in diameter carries a current of 5.0 A which is evenly distributed over its cross-section. Find the magnetic field strength

(a) at a point 5.0 mm from the surface of the wire;

(b) at the surface of the wire;

(c) at a point 2.5 mm from the axis of the wire.

A112. Whirly Tubes. Get out your whirly tube and give it a whirl. Observe it is possible to get different notes from it, but not any frequency you want; only a certain frequencies seem to be allowed. Let's understand where those are coming from.

Your whirly tube has a length of 75 cm and is open at both ends.

(a) Sketch the standing wave pattern for the three longest wavelength modes that fit these end conditions.

(b) From each sketch, determine the corresponding wavelength λ.

(c) Given your answers for (a) and (b), and given the speed of sound in air (340 m/s), determine the frequency of each mode.

(d) Now let's check the values you calculated. Here is a link to a tone generator. You can also access the same site with this QR code:

On this site enter one of the frequencies you calculated. How does it compare with the sound made when you whirl the tube? Check all three of the frequencies this way.

(e) The sound made by the whirly tube may not perfectly matching the calculated frequencies. Some questions: (a) Is the lowest frequency note that you are hearing, in fact, the lowest possible mode? For some tubes, the lowest note that you hear is actually the **second** lowest mode. So, instead of hearing the lowest three (1, 2 and 3), you might be hearing modes 2, 3 and 4. (b) The antinodes don't occur exactly at the ends of the tube. Adjust the frequency of the online tone generator up or down until the match is better. What does this tell you? Are the anti-nodes separated by a distance greater 75 cm or less than 75 cm?

Chapter 1

Phasors, Phasor Diagrams, and Wave Interference

1.1 Introduction

No, they're not little hand-held devices that can stun and kill unsuspecting aliens. At least, those aren't the ones we're talking about here. Phasors provide a way to represent oscillating motion graphically, and once you understand them a little, they provide for rather intuitive and straightforward way to handle wave interference problems.

1.2 A Phasor Diagram for a Single Oscillation

Consider a simple oscillating system, like a mass on a spring moving back and forth on a frictionless horizontal surface. We can write an expression for the displacement of the mass as follows:

$$x(t) = A\cos(\omega t), \tag{1.1}$$

where A is the amplitude of the oscillation, and ω is the angular frequency in units of rad/s. That means that A is the value of the largest deviation

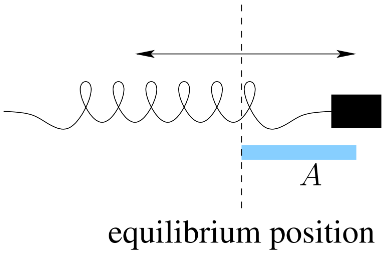


Figure 1.1: Mass on a spring.

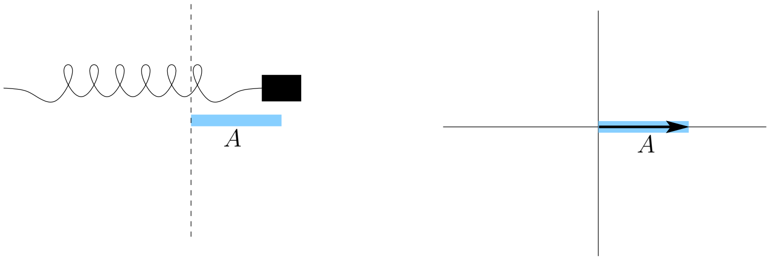


Figure 1.2: Oscillator and phasor at time $t = 0$.

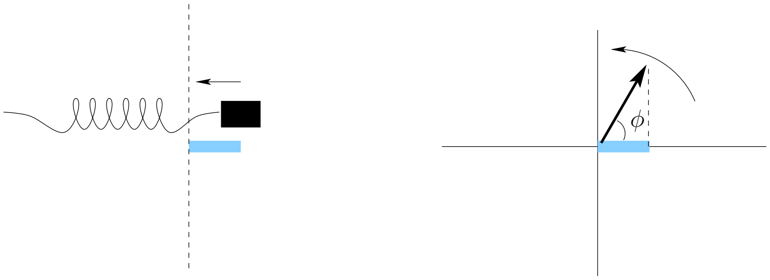


Figure 1.3: Oscillator and phasor a little bit later.

(plus or minus) of the mass from its equilibrium position, and the value of ω governs the time it takes for the mass to complete one oscillation (recall that the period of the oscillation, $T = 2\pi/\omega$, so a small ω means a long period while a large ω means a short period). Together, A and ω completely define the oscillation.

We can represent these two quantities, and therefore the oscillation itself, with a phasor diagram. A phasor is nothing more than a vector with magnitude A that *rotates* with angular velocity ω. The concept of a rotating vector is a little strange, but it works out really well for oscillations. For the oscillation defined above, $x(0) = A$ since $\cos(0) = 1$. We can represent the oscillation at this time with a phasor of length A that lies on the horizontal axis, as seen in Fig. 1.2.

Not very interesting yet, I know, but here's the cool part: let time advance. As time passes, the mass moves back toward its equilibrium position. At the same time, the phasor rotates in the counterclockwise direction. The phasor's magnitude doesn't change, but the projection of the phasor onto the horizontal axis (i.e., the horizontal component of the vector describing the phasor) gets smaller, as shown in Fig. 1.3.

Now if $\phi = \omega t$ in the phasor diagram, then the projection of the phasor on the horizontal axis is $A\cos(\phi) = A\cos(\omega t)$, and that's precisely the expression for the displacement of the mass at time t. This means that

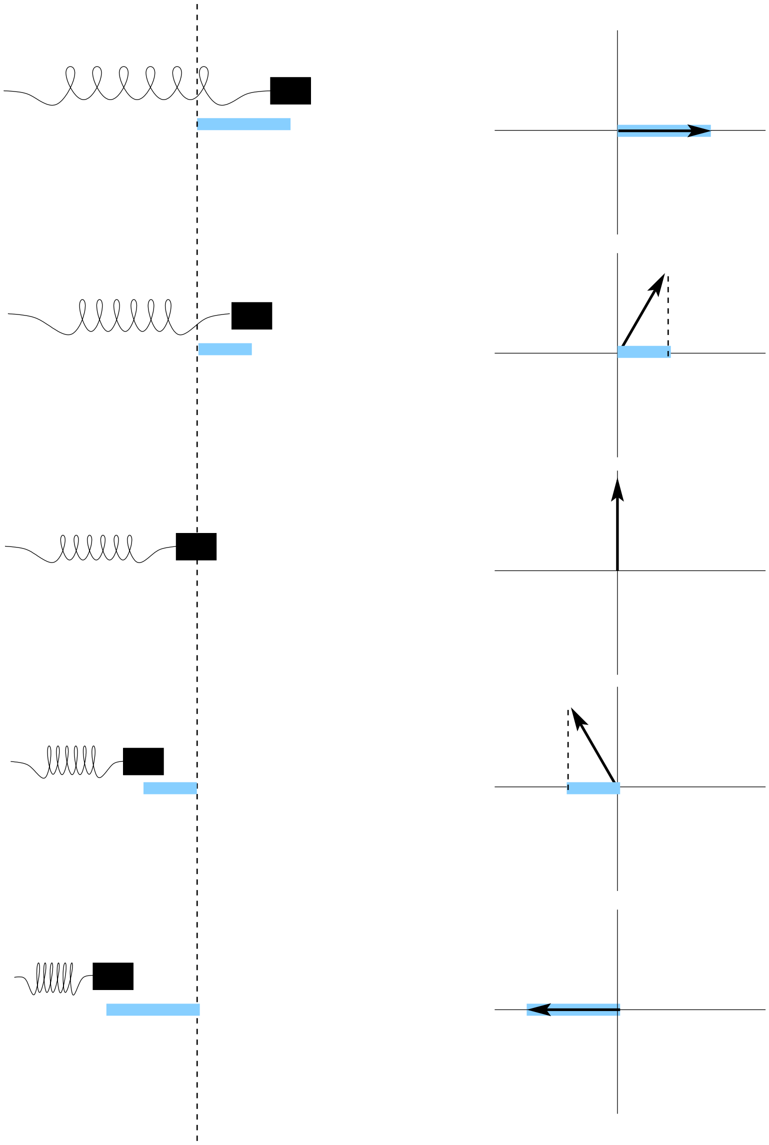


Figure 1.4: Time evolution of the oscillator and corresponding phasor.

if the phasor rotates with angular velocity ω and starts in the horizontal position at $t = 0$, its projection onto the horizontal axis will always describe the displacement of the oscillation, as seen in Fig. 1.4.

Note that the phasor does not change length; rather, it's just the projection of the phasor onto the horizontal axis that changes to describe the oscillation's changing displacement. This diagram should also give you a better idea of why we talk about "angular frequency" for oscillations. We're matching up the frequency of the oscillation with the angular velocity of the rotating phasor.

Example 1. A Day at the Beach.

Imagine you're at the beach and you walk into the surf until you're waist-deep in the water. You feel the waves passing by you, and you realize that you can express the change in the level of the water at your location with the expression

$$y(t) = A\cos(\omega t), \tag{1.2}$$

where $A = 20\,\text{cm}$ and $\omega = \frac{\pi}{3}\,\text{rad/s}$. Draw a phasor diagram to depict the level of the water at time $t = 3\,\text{s}$.

Solution: The phasor describing this oscillation will have magnitude $A = 20\,\text{cm}$, so we need a vector of that length. At time $t = 3\,\text{s}$, the phasor will have rotated so that

$$\phi(t) = \omega t = \frac{\pi}{3} \times 3 = \pi\,\text{rad}. \tag{1.3}$$

The phasor diagram is illustrated in Fig. 1.5. The projection of the phasor onto the horizontal axis is $-20\,\text{cm}$, so at this time, the wave height is at a minimum (i.e., a trough).

Important Note: Even though this problem is about an oscillation in the vertical direction, I still measure the displacement of the oscillation by the projection of the phasor onto the *horizontal* axis. There is no x- or y-axis on a phasor diagram, so don't go and try to match up the orientation of the oscillation with the phasor diagram.

Example 2. Alarm Bells.

A fire alarm goes off in your dorm. You run outside and stand in the cold weather waiting for the fire department to arrive and turn off the alarm (it's yet another false alarm). While you're waiting, you get out your pocket oscilloscope and measure the change in density of the air

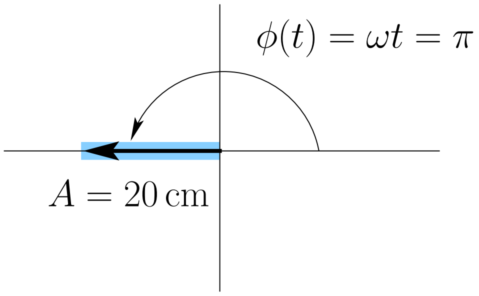


Figure 1.5: Phasor diagram for Example 1.

at your location due to the compression from the sound waves from the fire alarm. You measure the density change $\Delta\rho(t)$ to be described by the following expression:

$$\Delta\rho(t) = A\cos(\omega t + \phi_0), \tag{1.4}$$

where $A = 0.012\,\mathrm{kg/m^3}$, $\omega = 5000\pi\,\mathrm{rad/s}$, and $\phi_0 = \frac{\pi}{4}$. You decide to draw a phasor diagram describing the density at time $t = 0.1\,\mathrm{ms}$.

Solution: This problem is a little trickier because of the non-zero ϕ_0 inside the cosine. Conceptually, it means that when you decided to define time $t = 0$, the oscillation wasn't at a maximum. Instead, at time $t = 0$,

$$\Delta\rho(t) = A\cos(\phi_0) = 0.012\,\cos\left(\frac{\pi}{4}\right) = 0.0085\,\mathrm{kg/m^3}. \tag{1.5}$$

There's nothing wrong with this, but it will affect how we draw our phasor diagram. The amplitude of the oscillation is $A = 0.012\,\mathrm{kg/m^3}$, so that must be the length of our phasor. However, if the projection of

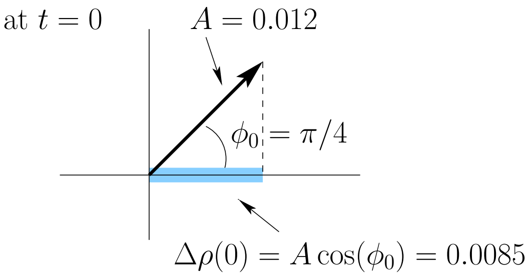


Figure 1.6: Phasor corresponding to an oscillator with a phase shift.

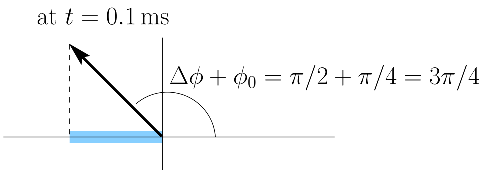


Figure 1.7: Phasor diagram of oscillator in Example 2 at $t = 0.1$

this phasor onto the horizontal axis at time $t = 0$ is to be $0.0085\,\mathrm{kg/m^3}$, then the phasor must be rotated. By how much? By ϕ_0 (see Fig. 1.6).

Fig. 1.6 is the "starting point" for our phasor diagram. To produce the phasor diagram for $t = 0.1\,\mathrm{ms}$, we need to let this phasor rotate for that time interval. It will rotate by

$$\Delta\phi(t) = \omega t = 5000\pi \times 0.0001 = \frac{\pi}{2} \tag{1.6}$$

from the starting point of $\phi_0 = \frac{\pi}{4}$. So our phasor diagram will look like Fig. 1.7.

The projection of the phasor at this time is

$$A \cos\left(\frac{3\pi}{4}\right) = -0.0085\,\mathrm{kg/m^3}. \tag{1.7}$$

That's the same value we get from the algebraic expression for $\Delta\rho(t)$:

$$\begin{aligned} \Delta\rho(t) &= A\cos(\omega t + \phi_0) \\ &= 0.012\,\cos\left(5000\pi \times 0.1 + \frac{\pi}{4}\right) \\ &= 0.012\,\cos\left(\frac{3\pi}{4}\right) \\ &= -0.0085\,\mathrm{kg/m^3}. \end{aligned} \tag{1.8}$$

The change in density at this moment is negative, so the density is lower than its equilibrium value due to the passing sound wave.

1.3 Adding Phasors

The power and utility of the phasor representation really show up when combining oscillations. Consider two oscillations, both with the same angular frequency ω, but with different amplitudes and phases:

$$y_1(t) = A_1 \cos(\omega t + \phi_1) \quad \text{and} \quad y_2(t) = A_2 \cos(\omega t + \phi_2). \tag{1.9}$$

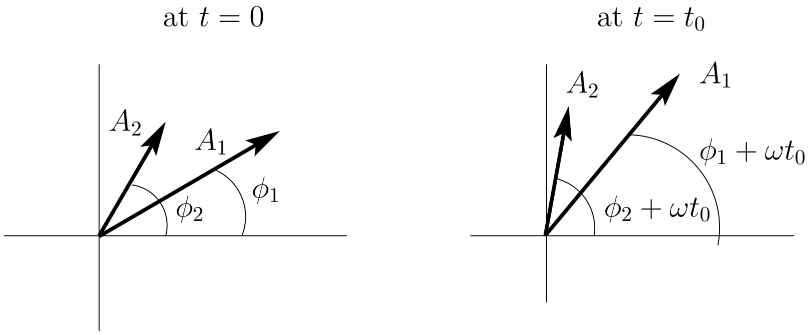


Figure 1.8: Phasor diagram showing separate phasors for oscillations y_1 and y_2 rotating together with the same angular frequency.

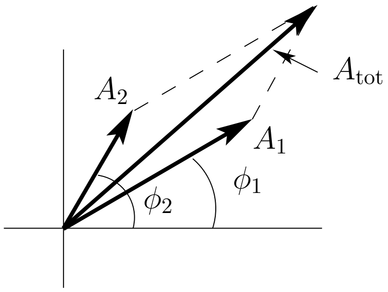


Figure 1.9: The combined oscillation is described by a phasor that is the vector sum of the two separate phasors.

The superposition of these two oscillations, $y_{\text{tot}} = y_1 + y_2$, will be sinusoidal with the same angular frequency ω, but it is very messy to calculate the new amplitude and phase shift algebraically; it is actually much simpler using the complex phasor method. Because the oscillation's have the same frequency, the phasors rotate together as time passes, as illustrated in Fig. 1.8. That is, the angle between the phasors will always be $\Delta\phi = \phi_2 - \phi_1$.

The resultant phasor, with its own amplitude and phase, is illustrated in Fig. 1.9. We can determine the amplitude and phase of the resultant phasor by adding the two input phasors the same way we add vectors. The amplitude of the resultant will be less than the sum of the two original phasor amplitudes (unless $\Delta\phi = 0$) and the phase of the resultant will be something between the phases of the two original phasors.

Example 3. Rock Your Boat.

You're sitting in a boat in the middle of a calm lake. Suddenly a motor boat drives by, producing waves that would oscillate your boat

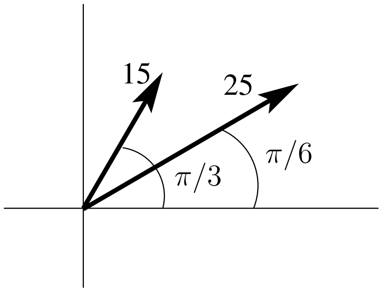


Figure 1.10: Phasor diagram for two water waves in Example 3.

up and down as follows:

$$y_1(t) = A_1 \cos(\omega t + \phi_1), \tag{1.10}$$

where $A_1 = 25\,\text{cm}$, $\omega = \frac{2\pi}{3}$, and $\phi_1 = \frac{\pi}{6}$. At the same time, another motor boat drives by, producing waves that would oscillate your boat up and down as follows:

$$y_2(t) = A_2 \cos(\omega t + \phi_2), \tag{1.11}$$

where $A_2 = 15\,\text{cm}$, $\omega = \frac{2\pi}{3}$, and $\phi_1 = \frac{\pi}{3}$. However, since both waves act on the boat simultaneously, the actual oscillation of your boat is the superposition of these two waves. How much does your boat move up and down as a result of the combination of these waves?

Solution: The phasor diagram for these two separate oscillations is shown in Fig. 1.10. The resultant phasor can be determined from the vector addition of the phasors shown in Fig. 1.11.

Phasor	real part (horizontal)	imaginary part (vertical)
1	$25\cos\left(\frac{\pi}{6}\right) = 21.6$	$25\sin\left(\frac{\pi}{6}\right) = 12.5$
2	$15\cos\left(\frac{\pi}{3}\right) = 7.5$	$15\sin\left(\frac{\pi}{3}\right) = 13.0$
Total	29.1	25.5

So, the amplitude of the resultant phasor is

$$A_{\text{tot}} = \sqrt{29.1^2 + 25.5^2} = 38.7\,\text{cm}, \tag{1.12}$$

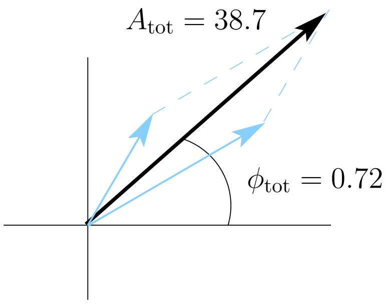


Figure 1.11: Phasor addition for two water waves in Example 3.

and its initial phase is

$$\phi_{\text{tot}} = \tan^{-1}\left(\frac{25.5}{29.1}\right) = 0.72\,\text{rad}. \tag{1.13}$$

We can write the superposition as

$$\begin{aligned} y_{\text{tot}}(t) &= A_{\text{tot}}\cos\left(\omega t + \phi_{\text{tot}}\right) \\ &= 38.7\cos\left(\omega t + 0.72\right), \end{aligned}$$

with y in cm and t in sec.

Example 4. Two Towers.

Two radio towers, A and B, separated by 20 m, broadcast the same radio signal of wavelength $\lambda = 12\,\text{m}$. You're standing at the point P indicated in the figure with your radio wave amplitude meter. With only tower A broadcasting, you measure a wave amplitude of 7 (in some unspecified unit). With only tower B broadcasting, you measure a wave amplitude of 5. What amplitude do you measure when both towers are broadcasting?

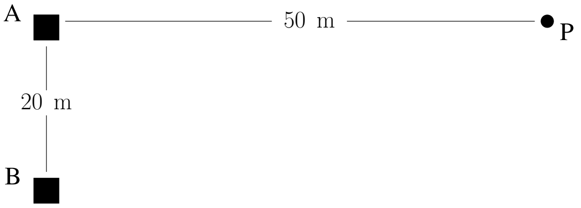


Figure 1.12: Illustration of radio towers in Example 4.

Solution: Here we're interested in the superposition of the waves from the two towers. We aren't explicitly given the phase difference between the wave signals that arrive at P from the two towers, but we can figure it out because we know the distances from point P to each tower. If the waves leave the two towers in phase, they won't necessarily be in phase when they reach point P because the waves from Tower B have to travel farther. How much farther? Well, the distance between Tower B and point P is $\sqrt{50^2+20^2} = 53.85\,\mathrm{m}$, so the waves from Tower B have to travel 3.85 m farther. That means the waves from Tower A will be 3.85 m ahead of the waves from Tower B.

We can turn that distance difference into a phase difference for the waves. The path length difference $\Delta r = 3.85\,\mathrm{m}$ produces $\frac{3.85}{12} = 0.32$ *wavelengths* of difference between the two waves as they arrive at point P. If the waves from Tower A arrived exactly one wavelength ahead of the waves from Tower B, the signals would add constructively. The phase difference would be 2π, and peaks and troughs of the two waves would be aligned. In this case, however, the waves from Tower A arrive only 0.32 wavelengths ahead of the waves from Tower B. Consequently, the phase difference $\Delta\phi$ is

$$\Delta\phi = 2\pi\frac{\Delta r}{\lambda} = 2\pi\frac{3.85}{12} = 2.0\,\mathrm{rad}. \tag{1.14}$$

We can then write expressions for the two oscillations at point P as follows:

$$y_A(t) = 7\cos(\omega t + 2.0) \qquad y_B(t) = 5\cos(\omega t). \tag{1.15}$$

Why did I pick zero initial phase for the signal from Tower B? Because I could. In this case (and in most cases we'll deal with), we're only interested in the phase difference between signals. Therefore, I can always choose to describe one wave with zero initial phase, and then put the phase difference into the expression for the other wave. With one phasor on the horizontal axis, the phasor addition is just easier.

Now we can construct a phasor diagram for the two oscillations, shown in Fig. 1.13.

Once again, the resultant phasor can be determined from the vector addition of the phasors.

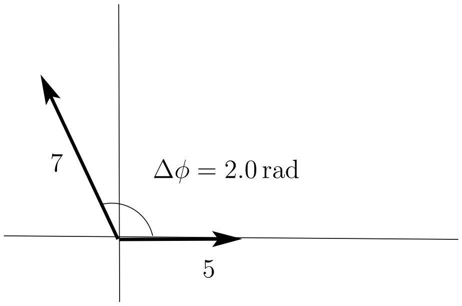


Figure 1.13: Phasor diagram for radio waves in Example 4.

Phasor	real part (horizontal)	imaginary part (vertical)
1	$5\cos(0) = 5$	$5\sin(0) = 0$
2	$7\cos(2.0) = -2.9$	$7\sin(2.0) = 6.4$
Total	2.1	6.4

So, the amplitude of the resultant phasor is

$$A_{\mathrm{tot}} = \sqrt{2.1^2 + 6.4^2} = 6.7, \tag{1.16}$$

and its initial phase is

$$\phi_{\mathrm{tot}} = \tan^{-1}\left(\frac{6.4}{2.1}\right) = 1.25\,\mathrm{rad}, \tag{1.17}$$

and we can thus write the superposed oscillation as

$$y_{\mathrm{tot}}(t) = 6.7\cos(\omega t + 1.25). \tag{1.18}$$

1.4 Phasor Diagrams for Multiple Source Interference

Sometimes, more than two waves will interfere at a single point. In these cases, the total amplitude of the combined oscillation will be the superposition of all of the incident waves. The phasor addition technique can be used to calculate the total amplitude in these complicated cases. Consider, for

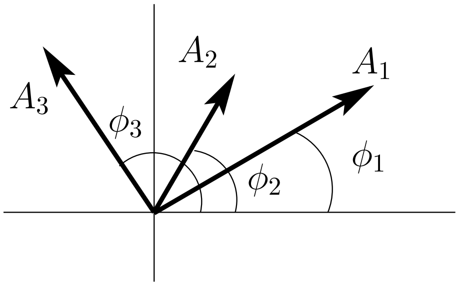


Figure 1.14: Phasors for the three oscillations y_1, y_2, and y_3.

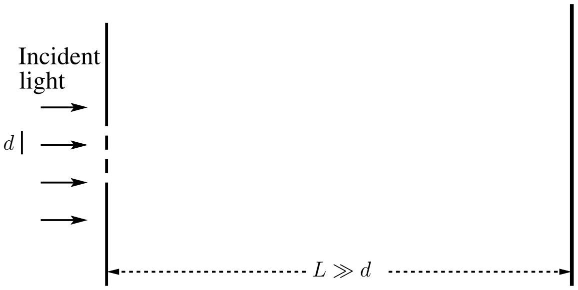


Figure 1.15: Three closely spaced slits, illuminated straight-on from the right. The interference pattern is observed on a distant screen to the right.

example, three oscillations, all with the same angular frequency ω, but with different amplitudes and phases:

$$
\begin{aligned}
y_1(t) &= A_1 \cos(\omega t + \phi_1), \\
y_2(t) &= A_2 \cos(\omega t + \phi_2), \text{ and} \\
y_3(t) &= A_3 \cos(\omega t + \phi_3).
\end{aligned}
$$

These three oscillations can be expressed graphically, as shown in Fig. 1.14, and their superposition, $y_{\text{tot}} = y_1 + y_2 + y_3$, can be determined from vector addition of the three phasors.

One of the most common multi-source interference problems is an extension of the two-slit problem considered in class. Imagine a light source passing through a screen containing not two but *three* slits, each separated from its neighbor by a distance d, as shown in Fig. 1.15. The interference pattern for this arrangement of slits will look different from the interference pattern for a two-slit setup.

If the interference pattern illuminates a screen a distance L from the slits, where $L \gg d$, then we can assume that the light rays from each of the slits to any particular point on the screen are parallel. (This is the same

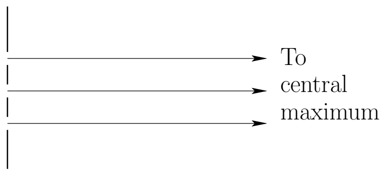


Figure 1.16: Light rays exiting the three slits parallel to the normal.

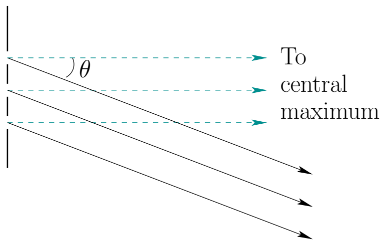


Figure 1.17: Light rays exiting at an angle θ to the normal.

approximation we made in the two-slit case.) For example, light passing straight through the slits and continuing straight to the screen produce the central maximum in the interference pattern, as in Fig. 1.16. These rays combine constructively because there's no difference in the path length for the three light rays, i.e., $\Delta r = 0$ for all three rays.

Now consider the light rays traveling at an angle θ to the normal, as in Fig. 1.17. Just as in the two-slit case, these rays do not travel the same distance to reach the screen. Light from the bottom slit travels the shortest distance, while light from the middle slit travels an extra distance Δr_{21}, and light from the top slit travels the same amount of extra distance ($\Delta r_{32} = \Delta r_{21}$) compared to light from the middle slit (Here, I've implicitly labeled the slits, #1, 2, and 3 for bottom, middle, and top.).

From the geometry of the problem (shown in Fig. 1.18), it's clear that $\Delta r_{32} = \Delta r_{21} = d \sin\theta$. We can then calculate the phase difference between light from adjacent slits,

$$\Delta\phi = 2\pi \frac{\Delta r}{\lambda} = 2\pi \frac{d \sin\theta}{\lambda}. \tag{1.19}$$

With knowledge of the phase difference, we can draw a phasor diagram and plot phasors for all three light rays, as shown in Fig. 1.19. We've assumed that the amplitudes of all three phasors is the same, and connected the phasors head-to-tail instead of placing all three with tails at the origin. Note that $\Delta\phi$ is the phase difference between *adjacent* phasors. The total amplitude of the combined rays is just the vector sum of the phasors:

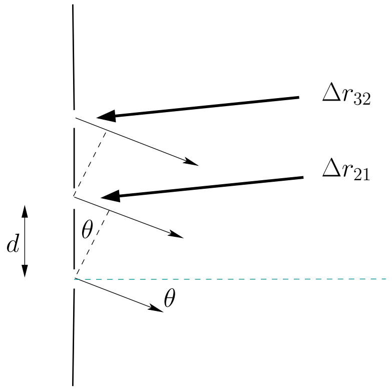


Figure 1.18: Geometry of path-length difference between rays from three slits. As for the double slit, the path length difference for adjacent slits is $d \sin \theta$ if the rays are nearly parallel.

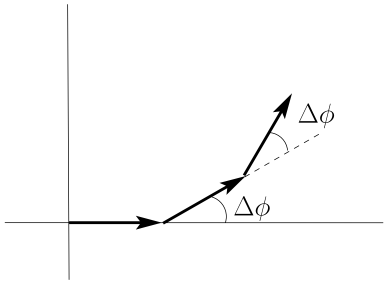


Figure 1.19: Three phasors, with adjacent phase difference $\Delta\phi_{\text{adj.}}$.

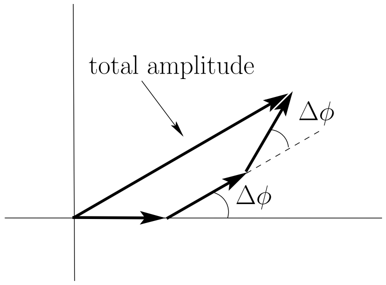


Figure 1.20: The resultant phasor is the vector sum of the three phasors corresponding to waves from the three slits.

Example 5. Combining Three Beams.

Light of wavelength $\lambda = 478\,\mathrm{nm}$ passes through a three-slit system where the slits are separated by a distance $d = 1.5 \times 10^{-6}\,\mathrm{m}$, and produces an interference pattern on a distant screen. What is the amplitude of the light directed at an angle $\theta = 1.2^\circ$ from the direction to the central maximum in the interference pattern? Assume that the amplitude of the light from each slit individually is A.

Solution: This is a direct application of the method described above. The phase difference between adjacent slits is

$$\begin{aligned}
\Delta\phi &= 2\pi\frac{\Delta r}{\lambda} \\
&= 2\pi\frac{d\sin\theta}{\lambda} \\
&= 2\pi\frac{1.5\times10^{-6}\times\sin(1.2^\circ)}{478\times10^{-9}} \\
&= 0.41\,\mathrm{rad}.
\end{aligned}$$

Therefore, our phasor diagram looks like Fig. 1.21. To calculate the total amplitude, we again add the phasors as vectors:

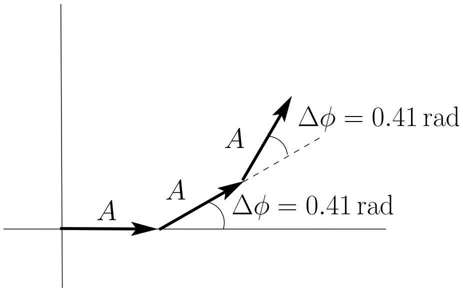


Figure 1.21: Phasor diagram for Example 5.

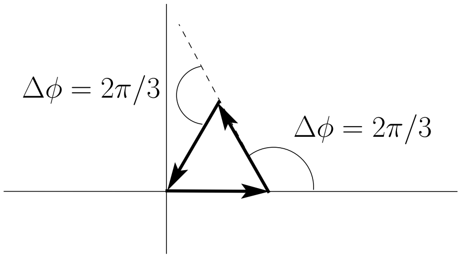


Figure 1.22: Phasor diagram at the first minimum of the three-slit interference pattern.

Phasor	real part (horizontal)	imaginary part (vertical)
1	$A\cos(0) = A$	$A\sin(0) = 0$
2	$A\cos(0.41) = 0.92A$	$A\sin(0.41) = 0.40A$
3	$A\cos(0.82) = 0.68A$	$A\sin(0.82) = 0.73A$
Total	$2.6A$	$1.13A$

So the amplitude of the resultant phasor is

$$A_{\text{tot}} = \sqrt{(2.6A)^2 + (1.13A)^2} = 2.83A. \tag{1.20}$$

Example 6. Minimum Light.

For the same setup as in the previous example, at what angle θ would you expect to find the first minimum?

Solution: A minimum in the interference pattern will occur when the amplitude of the resultant phasor is zero, so we need to arrange the three phasors such that they add to zero. When we had only two phasors, this was quite easy – we chose $\Delta\phi = \pi$. In this case, however, we need to make *three* phasors add to zero, and $\Delta\phi = \pi$ won't work (what would a phasor diagram with three phasors, each differing in phase from its neighbor by $\Delta\phi = \pi$ look like?)

To get three phasors to add to zero, we need a diagram like Fig. 1.22, where the phase difference between adjacent phasors is $\Delta\phi = \frac{2\pi}{3}$ rad. Make sure you understand why this arrangement yields a total amplitude of zero.

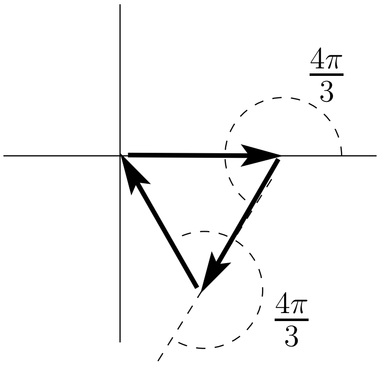


Figure 1.23: Phasor diagram at second minimum of the three-slit interference pattern.

Now that we know the required value for $\Delta\phi$, we can relate this to the path length difference between adjacent slits:

$$\Delta r = \frac{\Delta\phi}{2\pi}\lambda. \tag{1.21}$$

Since $\Delta r = d\sin\theta$, we can solve for $\sin\theta$:

$$\begin{aligned}\sin\theta &= \frac{\Delta r}{d} \\ &= \frac{\Delta\phi\lambda}{2\pi d} \\ &= \frac{2\pi(478\times10^{-9})}{6\pi(1.5\times10^{-6})} \\ &= 0.106\end{aligned}$$

and therefore $\theta = \sin^{-1}(0.106) = 6.1°$.

There's actually another way to make three phasors add to zero, and the phasor diagram looks like Fig. 1.23: In this case, $\Delta\phi = \frac{4\pi}{3}$, and you can show that this minimum will occur for $\theta = 12.3°$.

1.5 Single-Slit Diffraction

We've been assuming that all the slits used to produce diffraction patterns were very narrow. This meant that the distance from any point on the slit to a given point on the distant screen was the same. But what if we have a wide slit? (By wide, we mean a slit whose width a is comparable to or larger than the wavelength λ, but still very small compared with the distance to

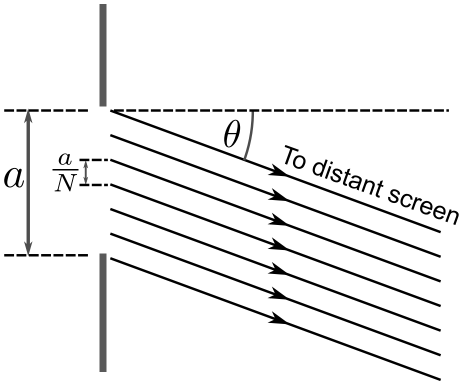


Figure 1.24: Subdividing a single slit of width a into N pieces of width a/N.

the screen L.) For such a slit the light from different parts of the slit may travel different distances on its way to the screen, and end up with different phase shifts. This seems like a complicated problem, but fortunately our phasor techniques are up to the job.

Here's the basic idea: we'll divide the wide slit up into a bunch of narrow sub-slits. We'll calculate the phasor corresponding to the wave from each sub-slit. And we'll add the phasors together to get a total amplitude.

Specifically, let's call the width of the slit a, and divide the slit into N pieces of width a/N as shown in Fig. 1.24. We'll keep N a variable, since we may want to divide the slit into 100 pieces, or 1000, or 10,000, or even take the limit $N \to \infty$. The phase difference between any two of our adjacent sub-slits is

$$\Delta\phi_{\text{adj.}} = \frac{2\pi\Delta r_{\text{adj.}}}{\lambda}. \tag{1.22}$$

And since the distance between two adjacent sub-slits is $d = a/N$, the adjacent path-length difference is

$$\Delta r_{\text{adj.}} = d\sin\theta = \frac{a}{N}\sin\theta, \tag{1.23}$$

and the adjacent phase difference is

$$\Delta\phi_{\text{adj.}} = \frac{2\pi a}{N\lambda}\sin\theta. \tag{1.24}$$

We'll assume each sub-slit is equally illuminated, so the out-going wave from each sub-slit has the same amplitude A. (Of course, as we divide the slit into more pieces, the light power in each piece will go down, so A will decrease. But for a fixed value of N the amplitude A is just a constant.)

We can now draw our phasor diagram. It consists of N equal-length phasors, each rotated by $\Delta\phi_{\text{adj.}} = \frac{2\pi a}{N\lambda}\sin\theta$ relative to its neighbor, as seen

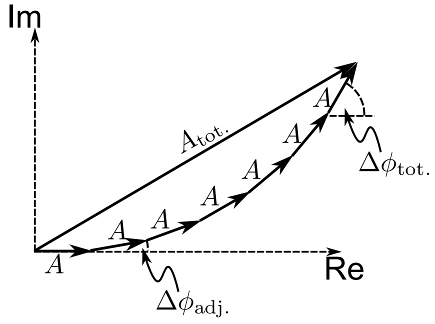


Figure 1.25: General phasor diagram for the single-slit diffraction pattern.

in Fig. 1.25. The final phasor is rotated by N times $\Delta\phi_{\text{adj.}}$, or

$$\Delta\phi_{\text{tot.}} = \frac{2\pi a}{\lambda}\sin\theta. \tag{1.25}$$

(Note that this is the same as the phase difference between the phasor from the top of the slit and phasor from the bottom of the slit.) Since the lengths and angles in our phasor diagram are all equal, the phasors together constitute a piece of a regular polygon. Now imagine making N big. For large enough N, the regular polygon is indistinguishable from a circle. So our phasor diagram is really just a circular arc.

Now we have all the information we need to calculate the amplitude of the diffraction pattern at some point on the screen. Straight ahead at $\theta = 0$, the total phase difference is $\Delta\phi_{\text{tot.}} = 0$, so the phasor diagram is just a straight line, with all the phasors pointing in the same direction, as shown in Fig. 1.26. As for our earlier double- and triple-slit interference patterns, this must be a maximum, since our chain of N phasors of length A is stretched as far from the origin as it can get, i.e. NA. If we move a certain distance away on the screen, to angular position θ, the phasor diagram becomes a circular arc with the same length (NA), starting horizontally at the origin and ending with its tangent at an angle $\Delta\phi_{\text{tot}} = 2\pi a\sin\theta/\lambda$. The resultant phasor is the chord of the circle, stretching from the origin to the ending point. By looking at Fig. 1.25 we can see that the amplitude is smaller now, since the phasors are no longer all pointing in the same direction. By adding these phasors and doing some geometry, we can now figure out how bright the pattern is anywhere on the screen.

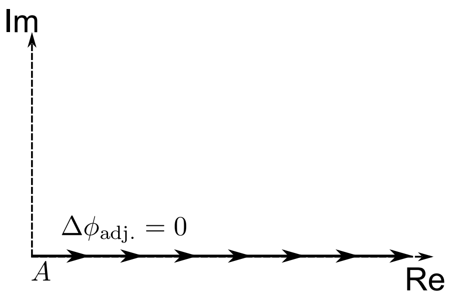


Figure 1.26: Phasor diagram for the maximum of the single-slit diffraction pattern.

Example 7. Minimum of Single-Slit Diffraction Pattern

Light of wavelength 600 nm illuminates a slit of width 2 μm. Find the distance between the central maximum and the neighboring minimum on a screen 1.5 m away.

Solution: We want to find a minimum of the diffraction pattern, i.e., a point at which the total amplitude goes to zero. Then the starting point of our N-phasor diagram must be the same as the ending point, so that the resultant has length zero. How do we make a circular arc end up where it started? By closing the circle, as shown in Fig. 1.27. From the diagram, we can read off the phase difference $\Delta\phi_{\text{tot}}$ between the first and last phasor–the first phasor points right, by the top of the circle it has rotated 180° or π radians to point left, and by the end, it has rotated around to 360° or 2π radians. Or, in other words, going in a complete circle means rotating by $\Delta\phi_{\text{tot}} = 2\pi$ radians.

Now we want to calculate where on the screen this minimum occurs. From here on out, it's the same calculation we've done with other slit patterns:

$$\begin{aligned}
\Delta\phi_{\text{tot}} = 2\pi &= \frac{2\pi a}{\lambda}\sin\theta \\
\lambda &= a\sin\theta \\
600\times 10^{-9}\,\text{m} &= 2\times 10^{-6}\,\text{m}\sin\theta \\
\theta &= \arcsin\left(\frac{6\times 10^{-7}}{2\times 10^{-6}}\right) = \arcsin(0.3) = 0.305\,\text{rad}
\end{aligned}$$

Now, by the geometry of the right triangle formed by the line to the minimum on the screen, the line to the central maximum, and the

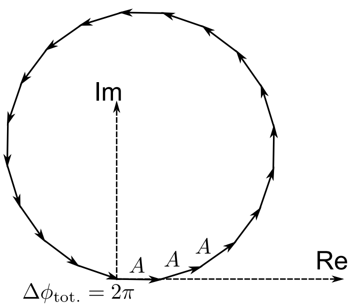


Figure 1.27: Phasor diagram for a minimum of the single-slit diffraction pattern.

screen itself, we can calculate the desired distance on the screen:

$$y = L\tan\theta$$

$$y = 1.5\,\mathrm{m}\tan(0.305) = \boxed{0.472\,\mathrm{m}}$$

1.6 Diffraction Limit

We can learn a lot from the single slit. As seen in Example 7, the first minimum of the single-slit pattern happens when $\Delta\phi_{\text{tot}} = 2\pi$ or $\lambda = a\sin\theta$. In many cases, the wavelength λ is substantially smaller than the size of the slit, and we can safely use the small-angle approximation $\sin\theta \approx \theta$. Note that θ must be in radians for this to be a good approximation–try it out on your calculator for some large (θ around 1 or bigger) and some small (θ much less than 1) values of θ, to convince yourself you understand how this

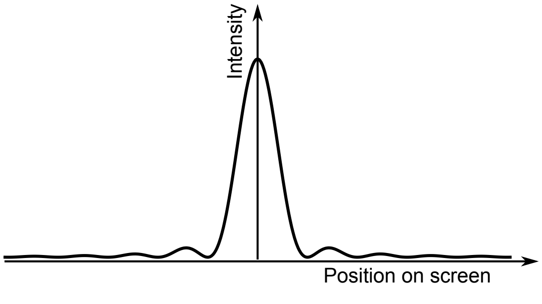


Figure 1.28: Single-slit diffraction pattern.

approximation works. In this approximation, we can then rearrange this equation:

$$\theta \approx \frac{\lambda}{a}. \tag{1.26}$$

This equation is known as the **diffraction limit**. It doesn't look like much, but we can squeeze a lot of useful information out of it. First of all, it says that when we decrease the wavelength λ, the waves spread out (diffract) by a smaller angle when they pass through a slit or aperture. Similarly, if we use a bigger slit, the waves also spread out less.

Why do we care how much wave spread out when they pass through an aperture? Because that's how we see and hear things, and how all optical instruments like telescopes, binoculars, and microscopes work.

Example 8. The Naked Eye

If your eye were limited solely by diffraction (i.e., your vision was perfect), would you be able to distinguish how many fingers your friend was holding up at a distance of 10 km? If not, how large a telescope would you need to distinguish them?

Solution: Let's estimate the distance between a person's fingers as about $\Delta x = 1\,\text{cm}$. The distance between you and your friend is $L = 10\,\text{km}$. So the angle between your friend's two fingers, as viewed by you, is

$$\theta_{\text{fingers}} = 2 \arctan \frac{\Delta x}{2L} = 2 \arctan \frac{1\,\text{cm}}{2 \cdot 10\,\text{km}} \approx 10^{-6}\,\text{rad}. \tag{1.27}$$

The diameter of your eye (the iris, or aperture) is around 0.5 cm, and let's assume we're using light in the middle of the visible spectrum, around 500 nm. So, using the diffraction limit equation above,

$$\theta_{\text{diffraction}} \approx \frac{500 \times 10^{-9}\,\text{m}}{0.5 \times 10^{-2}\,\text{m}} = 10^{-4}\,\text{rad}. \tag{1.28}$$

How can we interpret this? Well, we calculated that the angle between two light rays coming from the two adjacent fingers are only separated by an angle of $\theta_{\text{fingers}} = 10^{-6}\,\text{rad}$, but when they pass through they aperture of the eye, they diffract and are smeared out over an angle of about $\theta_{\text{diffraction}} = 10^{-4}\,\text{rad}$. As a result, the light from the two fingers is smeared into one large blurry blob, and you have no hope of distinguishing the two fingers. (In fact, you can do the calculation and show that at a distance of 10 km, you'd be unable to distinguish any two objects closer than about a meter with the naked eye.)

> What if we had a telescope? Then we'd effectively be able to make the diameter of the aperture bigger. How much bigger would a have to be for you to see your friends fingers? Well, our diffraction angle was a factor of 100 bigger than the angular separation of the fingers. So we'd have to drop $\theta_{\text{diffraction}}$ by a factor of 100. If we're not going to change λ, we'll have to increase the aperture by a factor of 100, which means using a telescope of diameter $a_{\text{telescope}} = 100a_{\text{eye}} \approx 0.5\,\text{m}$. That's a pretty hefty telescope — probably not one you'll want to carry around with you!

This example highlights the concept of **resolution**, which you may have encountered in other contexts, like photography or computer monitors. A high-resolution image is a very crisp, clear one, with minimal blurring. Since the two fingers in Example 8 were blurred together by diffraction, we say that they were not resolved.

Since blurring is a fairly fuzzy concept, we usually won't try to distinguish exactly when two objects go from being resolved to being unresolved. Mostly we'll be interested just in whether they're clearly resolved or not. As a side note, though, there is a clear criterion called Rayleigh's criterion that's sometimes used to state whether two objects are resolved or not. Rayleigh's criterion for determining the cutoff when two objects go from being resolved to unresolved is that when the maximum of one object's diffraction pattern lines up with the minimum of the other's diffraction pattern, then the objects go from being resolved to being unresolved.

The diffraction limit helps us understand many practical applications of waves. We've focused on making the aperture larger or smaller for light waves, but another option is to make λ smaller, either by using blue or ultraviolet light or by switching to electrons. It turns out, as we'll see when we study quantum mechanics, that electrons also have a wavelength, and for typical energies the electron wavelength is much shorter than an optical wavelength. Thus using electrons to image small objects can produce images with much higher resolution than is achievable with light. This is the fundamental principle behind the electron microscope. Going to shorter wavelengths (and higher frequencies) is also the basis of medical ultrasound measurements, which use very high-frequency sound waves to probe the internal structure of the human body. Using regular low-frequency sound waves would result extremely blurry images, since the diffraction angle would be large.

1.7 Diffraction Gratings

A diffraction grating consists of many very small, equally spaced slits. A grating can be either a transmission grating (as pictured in Fig. 1.29, where

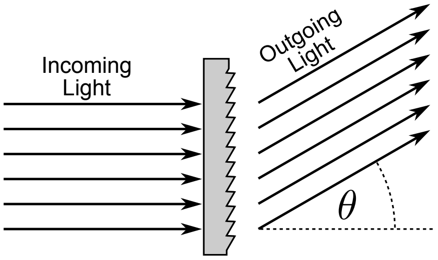


Figure 1.29: Light incident on a diffraction grating.

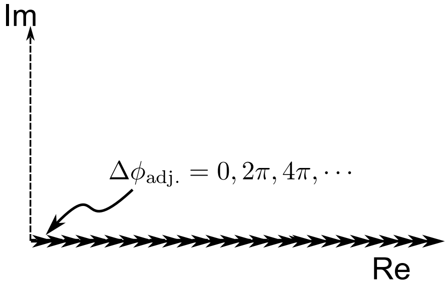


Figure 1.30: Phasor diagram for a diffraction grating at a maximum of the intensity.

the light comes from one side and gets diffracted out the other, or a reflection grating, where the light gets diffracted back on the same side of the grating that it came from. The “slits” may consist of absorbing and transparent regions, as with our double and triple slits, or simply of ridges in a transparent material, or of angled reflecting surfaces. Historically, the first diffraction gratings were made by painstakingly etching thousands of microscopic parallel grooves in glass using a diamond scribe. Fortunately, the details of how the grooves, lines, or slits are shaped doesn’t affect where the light gets diffracted. (It does affect how *much* light gets diffracted, but we won’t worry about calculating that.)

Everything we did for two and three slits carries over to a diffraction grating, except that now we have a very large number of phasors (in the range of 1000 to 10,000). That means we can’t easily draw them all and add them as vectors, but we can still figure out what the diffraction pattern must look like.

Suppose our grating is illuminated at normal incidence by light of wave-

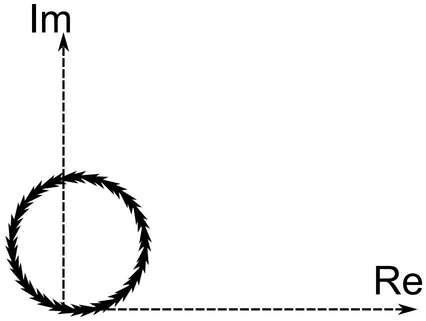


Figure 1.31: Phasor diagram for a diffraction grating away from a maximum.

length λ, so that the light reaches all the slits in phase. Each slit re-radiates its own outgoing wave, and these waves interfere on a distant screen. We represent each interfering wave as a phasor on a phasor diagram. To get a maximum from our many-phasor diagram, we want all the phasors pointing in the same direction, as in Fig. 1.30. This happens when the phase difference between two adjacent phasors is 0, but also when it is 2π, 4π, etc., and when it is -2π, -4π, etc. Thus we have maxima in the intensity for any integer multiple of 2π, provided the angle of the outgoing light is small enough that the light can actually reach the screen. These maxima are often referred to as “orders” of diffraction, where $\Delta\phi = 0$ is called the zeroth order, $\Delta\phi = 2\pi$ is the first order, $\Delta\phi = 4\pi$ is the second order, $\Delta\phi = -2\pi$ the minus-first order, etc.

What about points on the screen that *aren't* maxima? If we move away from the central maximum by a small distance, there's now some non-zero phase difference $\Delta\phi_{\text{adj.}}$. This means that each adjacent phasor is rotated by $\Delta\phi_{\text{adj.}}$ relative to its neighbor. But remember, we now have A LOT of phasors. So the by the time we've gotten through our 1000 or 10,000 phasors, the phasor diagram has wrapped itself up in a tight little circles, as indicated in Fig. 1.31. The amplitude won't be exactly zero, but it will be much less than the amplitude at the maximum. And the intensity, which is proportional to amplitude squared, will be MUCH, MUCH less than at the maximum. This is why the diffraction pattern from the grating consists of very narrow maxima, where all the light is concentrated, and wide expanses in between where essentially no light goes. (Incidentally, you can see from conservation of energy that when the maxima get very bright, they must also get very narrow–otherwise the total light power reaching the screen would be bigger than the light power passing through the slit!)

You can think of the grating diffraction pattern as the limit of the N-slit pattern, when N gets big. As shown in Fig. 1.32, the primary maxima of the multi-slit pattern get narrower and narrower as the number of slits increases (assuming a constant slit spacing). These maxima get correspond-

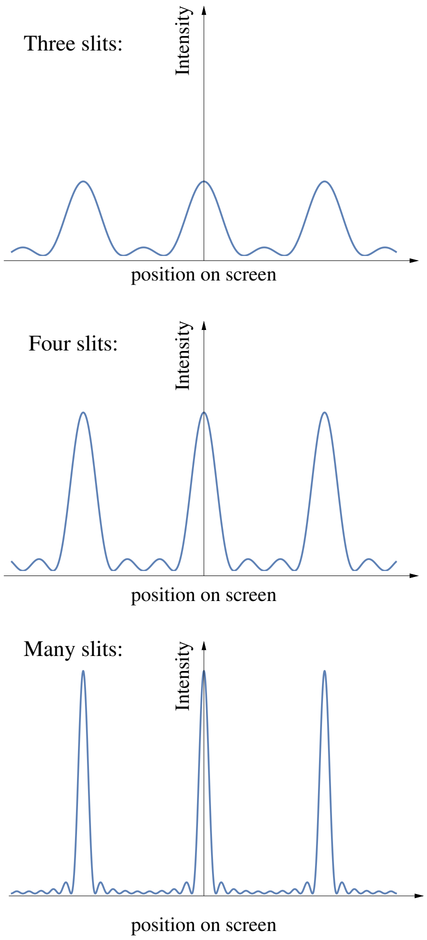


Figure 1.32: The diffraction pattern of multiple slits begins to resemble that of a diffraction grating as the number of slits increases, with very narrow, very bright peaks.

ingly brighter, since if the amplitude from each slit is A, the combined amplitude from N slits at a maximum is NA, and the combined intensity is proportional to N^2A^2, which grows rapidly as the number of slits increases.

Problems

1. Draw phasor diagrams depicting the oscillations described below at the noted times:

(a) For the oscillation $x(t) = 3\cos\left(\frac{\pi}{3}t\right)$, draw phasor diagrams for $t = 0$, 1, 2, 3, 5, and 6 s.

(b) For the oscillation

$$x(t) = 3\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right)$$

draw phasor diagrams for $t = 0$, 1, 2, 3, 5, and 6 s.

2. Consider water waves passing by a fixed point with a period of 6 s and a height 30 cm. Imagine that $t = 0$ corresponds to the moment when the water surface returns to its equilibrium position just after a crest passes.

(a) Draw phasor diagrams for the displacement of the water for times $t = 0, 1, 2, 3, 5$, and 6 s.

(b) Write an expression for this oscillation in the form

$$x(t) = A\cos\left(\omega t + \phi_0\right)$$

specifying A, ω, and ϕ_0.

3. Draw a phasor diagram describing these oscillations:

(a) $x(t) = 4\cos\left(\frac{\pi}{4}t + \frac{\pi}{2}\right)$ at time $t = 3$ s

(b) $x(t) = 4\cos\left(\frac{\pi}{2} - \frac{\pi}{4}t\right)$ at time $t = 3$ s

4. Write the superposition of the following two oscillations

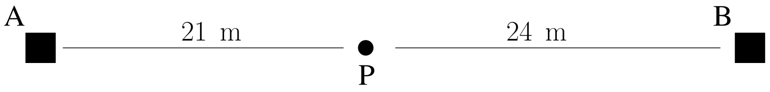
$$x_1(t) = 3\cos\left(\frac{\pi}{4}t\right) \qquad x_2(t) = 5\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$$

in the form

$$x_3(t) = A_3\cos\left(\omega t + \phi_3\right)$$

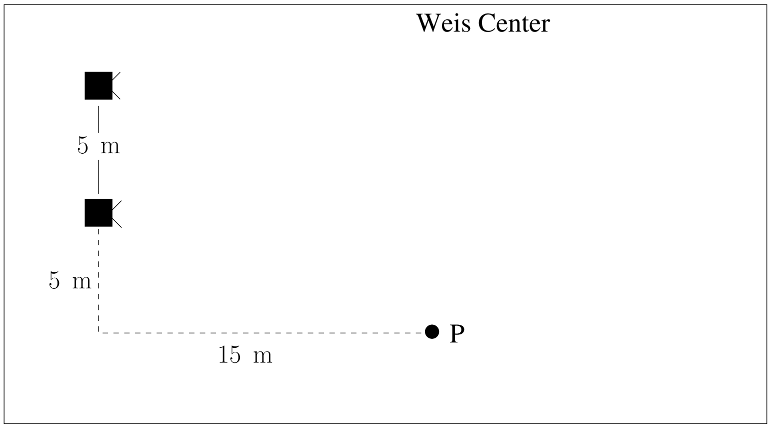
and determine A_3 and ϕ_3. (Hint: Use phasor addition!)

5. You're standing at the point P on a line between two radio towers, A and B.



Both towers broadcast the same radio signal of wavelength $\lambda = 12\,\mathrm{m}$. With only tower A broadcasting, you measure a wave amplitude of 6 (in some unspecified unit). With only tower B broadcasting, you measure a wave amplitude of 3. What amplitude do you measure when both towers are broadcasting?

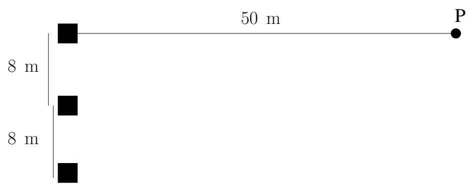
6. You're sitting in the Weis Center listening to the "Sonorous Symphony in C," which consists of a single 128 Hz tone played through two speakers separated by 5 m on stage. Both speakers emit sound waves in phase.



You happen to be sitting at the point P in the above diagram, 5 m to the left of the leftmost speaker, and 15 m back from the stage.

If the wave amplitude at your location from each speaker individually is A, what is the amplitude of the combined waves from both speakers at your location?

7. Three radio towers, arrayed as indicated in the figure, each broadcast the same in-phase signal with wavelength $\lambda = 5\,\mathrm{m}$.



If the amplitude from each tower individually at point P is A, what is the amplitude of the combined signal from the three towers?

(*Hint:* $d \sin\theta$ won't work here. Calculate the distances and figure out the path length differences directly.)

8. Light of wavelength $\lambda = 617\,\mathrm{nm}$ passes through a three-slit system where the slits are separated by a distance $d = 1.4 \times 10^{-5}\,\mathrm{m}$, and produces an interference pattern on a distant screen. Determine the amplitude of the light directed at an angle $\theta = 0.23^\circ$ from the direction to the central maximum in the interference pattern. Assume that the amplitude of the light from each slit individually is A.

9. Light of wavelength $\lambda = 532\,\mathrm{nm}$ passes through a four-slit system where the slits are separated by a distance $d = 1.7 \times 10^{-5}\,\mathrm{m}$, and produces an interference pattern on a distant screen.

(a) Draw phasor diagrams for the *three* minima between the central maximum and the first side maximum in the interference pattern, and determine the values of $\Delta\phi$ for each situation.

(b) Calculate the angles (relative to the direction to the central maximum) where each of these minima occurs.

10. Light of wavelength 600 nm is incident normally on a grating with 850 lines per millimeter. The diffraction pattern is observed on a distant screen.

(a) Sketch a phasor diagram for the light at the second-order maximum of the diffraction pattern. Be sure to indicate the phase difference between adjacent phasors on your diagram. (Don't worry about the fact that you can't draw all the thousands of phasors–just draw 5 or 6 representative phasors.)

(b) Using your phasor diagram and the adjacent phase difference you determined in part (a), find the angle to the normal at which the second-order diffraction maximum occurs.

11. Sodium has two emission lines with $\lambda_1 = 589.00\,\mathrm{nm}$and $\lambda_2 = 589.59\,\mathrm{nm}$. You send light made up of both wavelengths through a transmission grating with 1000 lines per millimeter and onto a screen one meter distant. By what distance will the first-order bright spots formed by the two wavelengths be separated?

12. You wish to design a spectrometer, using a transmission diffraction grating to spread light of different wavelengths out horizontally for observation on a distant screen. Your design specifications state that the second-order diffracted light must all fit on a screen 1.3 m away and 1.7 m wide, with the zero-order light hitting the center of the screen. The wavelengths you intend to analyze with the spectrometer fall between 450 nm and 800 nm.

(a) Determine the maximum allowed number of lines per millimeter for the grating.

(b) Using your answer for part (a), determine whether or not the first and second-order bands of diffracted light will overlap.

13. Light of wavelength 500 nm illuminates a single slit of width $5\,\mu\mathrm{m}$. At the center of the diffraction pattern on a screen 1 m away, the amplitude of the light is A. Determine the amplitude of the light on the screen a distance of 2.5 cm away from the center of the central maximum.

14. A telescope is being designed to detect distant binary-star systems. The telescope should be able to detect stars separated by 10^{-4} lt-yr at a distance of 300 lt-yr, using near infrared light of wavelength $1\,\mu\mathrm{m}$. Approximately what minimum diameter should the telescope have, if its performance is limited by diffraction?

Chapter 2

Beyond Classical Physics: Photons and Wave-Particle Duality

2.1 Introduction to Quantum Mechanics

In the early 1900s, a series of experiments and theoretical breakthroughs dramatically changed our understanding of how the universe works, including our conceptions of space and time, predictability versus randomness, and the limits imposed on measurement at the atomic scale. Not only did these developments — referred to as *modern physics* — significantly overturn and revise the known laws of physics, they also became the foundation for a complete understanding of the foundations of chemistry (and therefore biology as well), and they led to a series of significant applications that have already resulted in an explosion of modern technology, e.g., semiconductor physics and modern electronics; computer technology; communications (including cell phones); numerous medical diagnostic devices, surgical techniques and radiation therapies; new and significantly enhanced forms of microscopy; ultra-precise navigation devices; and nuclear power generation and weaponry.

And the technological applications of modern physics will continue well into the future. In particular, there is a significant on-going research effort into the development of nanotechnological devices that will likely revolutionize the fields of medicine and engineering during the next 50 years. Imagine, for instance, nanometer-scale devices that could be programmed to search and destroy cancer cells or repair internal injuries at the cellular scale within the human body. This idea is currently speculative (as of the year 2021), but successful development of something like this — which would almost certainly require an understanding of principles of modern physics — could revolutionize cancer treatment in a way that will cause society to look back at chemotherapy and radiation treatment the way we currently look back at the use of leeches as medical “devices” in the middle ages. Other potential future quantum applications include quantum computing and encryption; replacements for semiconductor switches based on new graphene materials or quantum optical devices; or even molecular and biological electronic devices based on DNA.

In PHYS 211, you learned about one of the two pillars of modern physics, Einstein’s Theory of Relativity, which extends classical (Newtonian) physics to systems that travel at speeds approaching the speed of light and also sheds important light on the nature of matter and energy. In this unit, you will learn about the other pillar of modern physics: the theory of quantum mechanics. Quantum effects are most noticeable on a microscopic scale, yet quantum behavior has critical effects on the macroscopic world that we see around us every day. As an example, atoms wouldn’t be possible without quantum principles, so all matter around us would be dramatically different without this subatomic behavior (actually, there wouldn’t be any “us”). And we have developed techniques that enable us to manipulate the subatomic world in ways that have significant technological applications, including much of what forms the basis of modern chemistry, chemical engineering, materials science, and electronics.

What we’ve learned about the microscopic world is that it is *unlike anything we can picture!* Things in the microscopic world, such as electrons or protons or electric fields, can sometimes act as particles and sometimes act as waves. Nothing in our macroscopic world is like this, so it is not easy to build simple mental pictures for it. To give you a sense of how strange this is: when a “particle” such as an electron is “acting as a wave” it does not have a definite position! Its very existence is spread out in some manner over a region of space.

So be it. It’s one of the greatest triumphs of science that we have developed the experimental and theoretical tools to uncover the behavior of nature even when our fundamental intuition can no longer aid us. The theory of quantum mechanics, as we shall see, is necessarily somewhat abstract. But it is very much a *physical* theory, well-grounded in experimental evidence.

To be an educated, scientifically aware citizen in the 21st century, it is essential to have a grounding in the basic laws of quantum mechanics.

2.2 Three Great Failures of Classical Physics

What emerged in the early years of the 20$^{\text{th}}$ century were puzzles and mysteries that our well-developed and successful theories of Newtonian mechanics, electricity and magnetism, and thermodynamics — what we now call *classical physics* — were unable to solve. Three of these mysteries stood out above the others.

The ultraviolet catastrophe. Combining classical electromagnetism with thermodynamics led to the (wrong) result that the thermal motion of matter — the acceleration of jiggling charges — would create an infinite amount of electromagnetic energy. Oops!

The stability of atoms. Combining classical electromagnetism (EM) with classical mechanics led to the result that electrons orbiting the nucleus of an atom would radiate away their energy and collapse into the nucleus after about 10^{-12} s. So atoms shouldn't be stable. Oops again!

Atomic spectral lines. Experiments showed that when energy is pumped into atoms, say by heating a gas, the atoms then radiate EM waves back out, but only at certain distinct wavelengths. Hydrogen atoms emit one set of wavelengths, helium atoms a different set, and so on. Nothing in classical physics could come close to explaining this phenomenon. Strike three!

The attempt to resolve these issues led to the revolutionary, paradigm-changing, development of quantum mechanics. In this unit we will show how a new quantum theory was able to address successfully these shortcomings of the classical theory.

The story begins with Planck and Einstein, who introduced the notion that EM waves, for example light waves, can also act as particles. This is a phenomenon called *wave-particle duality*: light is neither just a wave nor just particles, but has aspects of both. In this chapter we will look closer at the ultraviolet catastrophe — the incompatibility of thermodynamics and electromagnetism — and show that the particle aspect of light resolves the problem. Then we will explore the implications for the interaction between light and matter (explaining, for example, why you should wear sunscreen when you are outside for long periods on a sunny day). Finally, we conclude with de Broglie's stunning and ultimately correct hypothesis that not just light but all matter in the universe exhibits wave-particle duality.

2.3 Ultraviolet Catastrophe

Electromagnetism and thermodynamics were well-developed and successful theories by the end of the 19^{th} century. The theory of electromagnetism (Maxwell's equations) demonstrated the unification of electricity and magnetism, and led to the realization that light is an electromagnetic wave. This, in turn, led to the technology of generating and receiving radio waves, which was the second great step in the information technology revolution.[1] Similarly, our theories of thermodynamics explained the states of matter and provided an understanding of the engines that powered the industrial revolution.

---

[1] The printing press was the first.

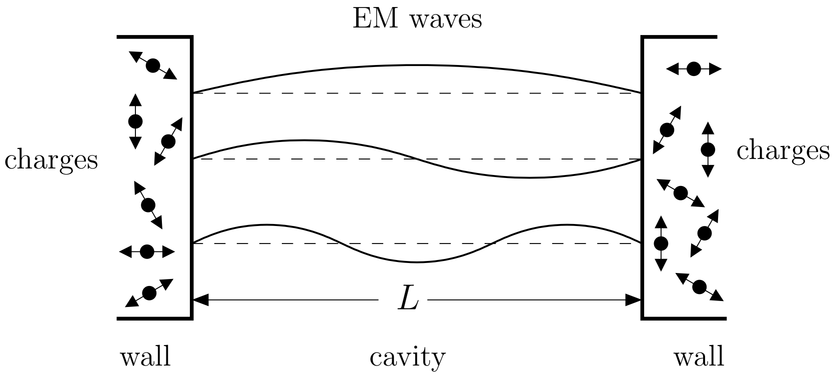


Figure 2.1: EM waves in cavity of length L. These EM waves are created by the thermal motion of the charges in the walls.

There was just one big problem: classical E&M and thermodynamics are incompatible. Here is the basic issue. Picture some substance at a temperature T. The particles in that substance are vibrating around with thermal motion. This amounts to accelerating charges, and accelerating charges emit EM waves. Conclusion: we expect matter at a temperature T to be radiating away some of its energy into electric and magnetic fields.

But the energy exchange goes both ways. These EM fields exert forces on the charged particles, giving them back some energy. So all together we see that energy sloshes between the moving particles and the EM fields, much like it sloshes from particle to particle within the substance. Then we should expect the EM fields, like the particle motion, to have some thermal equilibrium values determined by the temperature T.

So far, so good. Let us now try to calculate what the thermal equilibrium EM fields should be for the simplest case we can construct. Imagine a cavity bounded by a pair of walls separated by a distance L (i.e., a one-dimensional box), and to keep things simple we will only have one spatial dimension. The situation is illustrated in Fig. 2.1. The charges in the walls are in thermal equilibrium at temperature T, so they are moving and creating electric and magnetic fields. What form can these EM fields take? Basically, we get all the possible standing wave modes, just like waves on a string or sound waves in a tube. As we found in the waves unit, the longest wavelength is $\lambda = 2L$, the second longest $\lambda = L$, the third longest is $\lambda = 2L/3$, and so on. The actual EM fields contain all those modes and can be expressed as some superposition of these waves.

Now let's try to calculate the amplitude of these waves. First, recall that the energy of a wave is proportional to the amplitude squared. For a superposition of wave modes the energy turns out to be the superposition of the energies of each individual wave:

$$E = \alpha A_1^2 + \alpha A_2^2 + \alpha A_3^2 + \dots \tag{2.1}$$

where α is some constant and A_n is the amplitude of the n^{th} longest wavelength mode of the EM wave. Now comes the thermodynamics: the equipartition theorem says that in thermal equilibrium any quadratic term in the energy has an average value of $\frac{1}{2}k_BT$. Therefore we can conclude

$$\langle \alpha A_1^2 \rangle = \langle \alpha A_2^2 \rangle = \langle \alpha A_3^2 \rangle = \dots = \frac{1}{2}k_BT. \tag{2.2}$$

The equipartition theorem implies that each of those modes has to have the same average value, and they are all related to the temperature T. It's a powerful theorem!

And now we are ready to compute the total energy in the EM fields. All we need to do is count the number of modes in the cavity ($n = 1, 2, 3, \dots$) and then multiply by $\frac{1}{2}k_BT$. But here is the problem: *There are an infinite number of modes!*

We can keep drawing waves with shorter and shorter wavelength, and we never run out of modes. Each new one we draw brings another $\frac{1}{2}k_BT$ to the energy. So we are led to conclude that there is an infinite amount of energy in the EM fields. This is clearly wrong — thankfully, or we would all be blasted by the infinite radiation all around us. And now we can see why this is called the ultraviolet catastrophe: it is the infinite piling up of shorter and shorter wavelengths, or higher and higher frequencies, that is where something in the theories are breaking down.

So where did it go wrong?

2.4 Photons

The resolution of the ultraviolet catastrophe, reached in stages by Planck and Einstein, is that Maxwell's theory of electricity and magnetism is incomplete. In 1905[2] Einstein introduced the concept of a photon: a "particle" of light whose energy is related to the EM wave frequency via

$$E_{\text{ph}} = hf \tag{2.3}$$

where

$$h = 6.63 \times 10^{-34}\,\text{J}\cdot\text{s} = 4.14 \times 10^{-15}\,\text{eV}\cdot\text{s} \tag{2.4}$$

is known as Planck's constant. Einstein's claim was that an EM wave of frequency f can be viewed as a gas of photons, each with this energy. Photons

---

[2] 1905 was Einstein's "Miracle Year" — the year that he published his first paper on relativity, a paper that explained molecular diffusion, and a paper introducing the concept of a photon.

are not part of Maxwell's equations; they are in some sense a supplementary property of EM waves.[3]

Example 1. Number of photons.

An EM wave of wavelength $\lambda = 420\,\text{nm}$ has a total energy of 1.80 J. How many photons are in this wave?

Solution: Each photon is contributing an energy $E_{\text{ph}} = hf$. Using $f = c/\lambda$ we can calculate the energy per photon,

$$\begin{aligned} E_{\text{ph}} &= \frac{hc}{\lambda} \\ &= \frac{(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3.00\times10^{8}\,\text{m/s})}{420\times10^{-9}\,\text{m}} \\ &= 4.74\times10^{-19}\,\text{J}. \end{aligned} \tag{2.5}$$

The total energy E is given by NE_{ph}, where N is the number of photons, so

$$N = \frac{E}{E_{\text{ph}}} = \frac{1.80\,\text{J}}{4.74\times10^{-19}\,\text{J}} = 3.80\times10^{18}. \tag{2.6}$$

Evidently, EM waves with macroscopic energies contain a lot of photons!

The idea of light being composed of particle-like photons is tremendously important with numerous implications throughout all of science and engineering. The photon picture also resolves the problem of infinite energy in a cavity. Imagine sorting all the EM wave modes in the cavity from lowest frequency to highest frequency. Since $E_{\text{ph}} = hf$, the photons in the lowest frequency mode have low energy, while the photons in the high frequency modes have higher energy. But the total energy of each mode must be $\frac{1}{2}k_BT$, which means each EM wave mode has exactly the same energy.

How can higher energy photons add up to the same energy? There must be fewer of them. So as we go to higher and higher frequency modes, we must have fewer and fewer photons in each mode. Eventually we will reach a high frequency mode with only one photon, and that's the end of the line. There can be no higher frequency modes. And so the ultraviolet catastrophe is averted.

---

[3] The complete theory of electricity and magnetism, including the full reconciliation with quantum mechanics and relativity, is called *quantum electrodynamics* and wasn't fully developed until the 1950s. We will get a flavor of this theory in Unit 4.

In Problems 9 and 10 at the end of this chapter, you will do this quantitatively. You will find that the mode number j_{max} where the number of photons drops to below 1 in a one-dimensional cavity is

$$j_{\text{max}} = \frac{Lk_BT}{hc}. \tag{2.7}$$

For modes with larger j, there is not enough thermal energy to produce even a single photon. Given this result, we can find the total energy in the cavity. We have modes $j = 1, 2, \ldots, j_{\text{max}}$, each contributing an energy $\frac{1}{2}k_BT$. That adds up to

$$E = j_{\text{max}} \times \left(\frac{1}{2}k_BT\right) = \frac{L(k_BT)^2}{2hc}. \tag{2.8}$$

This answer is a much nicer result for the thermal energy of the EM fields, because it's not infinite!

Let's reflect a moment on what we have just done. We argued that EM waves, which in classical physics can have any continuous value of energy, instead come in energy chunks. When the number of these photon chunks is large, some 10^{18} or so, then we can ignore the chunkiness and treat the EM waves classically. Then the equipartition holds and we can use Eq. (2.2).

But for really high frequency (UV) waves, the photons have so much energy that there are only a few of them in a mode. At this point, the classical physics result Eq. (2.2) no longer holds. Essentially, there is simply not enough thermal energy available to create even a single photon for these modes. So we counted an energy of $\frac{1}{2}k_BT$ for each mode until we hit this wall, and then we stopped counting. The result is Eq. (2.8).

Note that this result only applies to a one-dimensional system. Real materials, of course, are three dimensional. We won't go through the full derivation for a three-dimensional cavity. That would require working with the thermodynamics of a photon gas, something that we do in our PHYS 317 Thermodynamics course.

2.5 Photons Interacting with Matter

In the previous section we argued why the photon picture resolved the dilemma of the UV catastrophe, but many physicists at the time that Einstein proposed photons were skeptical of the idea. But contained in Einstein's brilliant suggestion was the potential for a quantitative test of the idea, through a phenomenon called the photoelectric effect. This is the work for which he was awarded his only Nobel Prize.[4]

---

[4] Einstein should have received five Nobel Prizes, for special relativity, the photoelectric effect, Brownian motion (which demonstrated the existence of atoms), general relativity, and Bose-Einstein condensation.

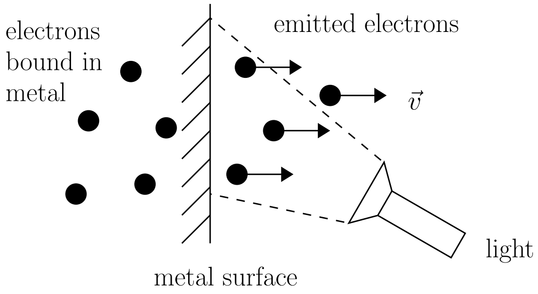


Figure 2.2: A sketch of the photoelectric effect: light shines on a metal, and electrons are released from the surface.

2.5.1 Photoelectric Effect

Metals are materials in which electrons are free to move around — this is why metals are good conductors, why they are shiny, etc. But the electrons cannot just freely hop off the surface. They are bound to remain inside the material by a binding energy U_{bind}. It was discovered at the close of the 19^{th} century that some electrons can be made to fly off the surface of a metal by shining light on it, as sketched in Fig. 2.2. This should seem reasonable: the EM wave gives some extra energy to the electrons, allowing them to break free of their binding to the metal. The electron "pays off" its binding energy debt and leaves with the remaining energy in the form of kinetic energy.

Let's focus on how this effect should depend on the intensity and the frequency of the light. With classical EM waves, increasing the amplitude of the wave should cause a stronger force on the electrons and give them more energy. This would suggest that increasing the intensity of light should increase the energy given to the electrons and they should come off with a higher final speed. Also, the classical EM wave picture would predict that it should be possible to eject electrons from the metal for any frequency of light, as long as the intensity is high enough.

By 1905 there were qualitative indications that this was not what was happening in experiments, and Einstein's photon hypothesis provided a quantitative theory that could be tested experimentally. In Einstein's theory, which was quantitatively verified by experiments in 1916, the intensity of the light has *no effect* on the kinetic energy of the emitted electrons — it is the frequency of the light that is related to the kinetic energy. Shining low frequency light on the material results in no electrons being emitted, regardless of the intensity, and when electrons are emitted, the kinetic energy increases linearly with the frequency of the light. The experimental results are illustrated graphically in the Fig. 2.3.

Einstein's theory of the photoelectric effect is based on the following key assumption:

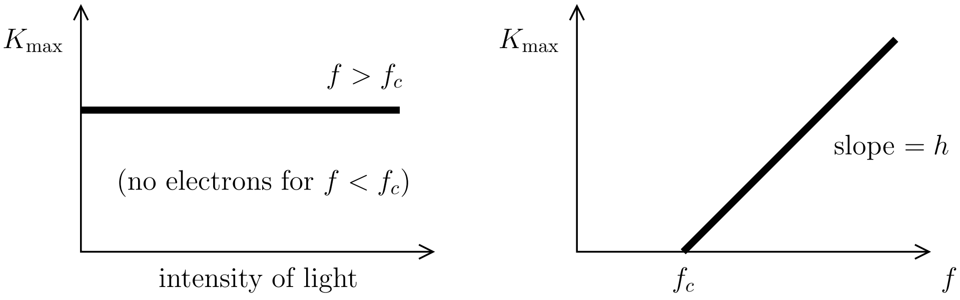


Figure 2.3: Left: the maximum kinetic energy of the emitted electrons is independent of the intensity of light. Also, electrons are not emitted regardless of the intensity of light if the frequency f is below some cutoff value f_c. Right: the maximum kinetic energy of the emitted electrons does depend on frequency, and the slope of the graph is equal to Planck's constant.

> In the microscopic world of atoms and subatomic particles, light interacts with matter in the form of *single photons*.

This means light of frequency f can only give a single photon's energy to an electron. That photon, with energy $E_{\text{ph}} = hf$, might or might not have enough energy to free the electron from the metal. If the photon energy is below the binding energy ($E_{\text{ph}} < U_{\text{bind}}$), then no electrons will escape. Since $E_{\text{ph}} = hf$, this sets the cutoff frequency f_c when $E_{\text{ph}} = hf_c = U_{\text{bind}}$. If the photon energy is greater than the binding energy holding the electron to the metal ($E_{\text{ph}} > U_{\text{bind}}$), then the electron might escape (it can always squander the energy and head off in the wrong direction, or bump into an impurity in the metal). But of those electrons that do escape, there will be an *upper limit* to their kinetic energy given by

$$\begin{aligned} K_{\max} &= E_{\text{ph}} - U_{\text{bind}} \\ &= hf - U_{\text{bind}}. \end{aligned} \tag{2.9}$$

That upper limit is obtained when the electron uses the energy absorbed from the photon optimally. Note that the maximum electron kinetic energy depends only on the frequency of the light and not on its intensity. This was Einstein's bold prediction, and it was confirmed experimentally by Millikan, who measured $K_{\max}$ for various frequencies and found data of the form sketched on the right in Fig. 2.3.

Thus we see that the photon picture explains the experimental results for the photoelectric effect perfectly. It also provides a quantitative measurement of Planck's constant h from the slope of the $K_{\max}$ versus f plot.

Example 2. Electrons emitted from copper.

Copper has a binding energy of 4.7 eV. For light of **(a)** 200 nm and **(b)** 400 nm shining on a piece of copper, determine whether electrons are emitted and, in the case they are emitted, calculate their maximum kinetic energy.

Solution: First we calculate the photon energy. A handy trick for getting photon energies in electron volts is the following:[5]

$$E_{\text{ph}} = hf = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{\lambda}. \tag{2.10}$$

For (a), $\lambda = 200\,\text{nm}$, so the photon energy is

$$E_{\text{ph}} = \frac{1240\,\text{eV}\cdot\text{nm}}{200\,\text{nm}} = 6.2\,\text{eV}. \tag{2.11}$$

When the photon is absorbed by the electron, the electron gains enough energy to pay off its binding energy debt and escape with

$$K_{\text{max}} = E_{\text{ph}} - U_{\text{bind}} = 6.2 - 4.7 = \boxed{1.5\,\text{eV}}. \tag{2.12}$$

For (b) we have a photon energy of $1240/400 = 3.1\,\text{eV}$. This is less than the binding energy, so no electrons will be emitted.

2.5.2 Ionization

Einstein's basic idea that photons are the energy chunks in the interaction of light with matter is able to explain more phenomena than just the photoelectric effect. For example, an electron in an atomic orbital is held in the atom by some binding energy. For the case of hydrogen — which is just a single electron orbiting a single proton — the electron is usually in the lowest energy state possible, called the ground state. In the hydrogen ground state the binding energy for the electron is 13.6 eV. When lower frequency radiation (EM waves) shines on a gas of hydrogen, then none of the electrons are freed from their atomic orbitals. Radiation of higher frequency is able to liberate electrons from their host nucleus, creating *ionized hydrogen*. This higher frequency radiation is called, rather sensibly, ionizing radiation.

This idea extends beyond simple hydrogen. EM waves frequently induce chemical reactions, from a photographic plate to photosynthesis to mutation of DNA. That is, the energy of the absorbed photon is being used to break

---

[5] The relation $hc = 1240\,\text{eV}\cdot\text{nm}$ can save a lot of calculator typing for problems with photon energies in eV and wavelengths in nm.

and possibly re-arrange some chemical bonds. In the case of DNA mutation, the relevant energy scale of the chemical bonds is in the 0.1 to 10 eV range; EM waves with photon energies comparable to or greater than these binding energies can cause mutations. **This is a known mechanism for explaining why exposure to EM radiation with certain frequencies can cause cancer in humans.** The photon nature of light is also critical for understanding various photo-detectors, including the sensors in the digital camera in your cell phone. We will explore these ideas in the assigned problems and in subsequent chapters.

Finally, we conclude this section with an important point: light had previously been considered to be exclusively a wave phenomenon, but with photons it has been shown to have particle characteristics. But it's not an either/or situation. It is not the case that the wave description is wrong; after all, you have seen the interference phenomena yourself in lab. So we are led to acknowledge that light can be both wave-like and particle-like. What it does in a particular experiment depends on what the experiment is measuring. This is very strange!

2.6 Wave-Particle Duality

Planck and Einstein made the first big steps towards quantum mechanics by introducing photons and showing the role they play when light interacts with matter. The next breakthrough was made by de Broglie (pronounced de-BROY), who proposed what is now called *wave-particle duality.*

Basically, de Broglie noticed how strange the wave and photon character of light is and pondered whether this was perhaps not limited to light, but rather was a new feature of nature. Since light, which had been considered a wave, could turn out to have particle properties, perhaps things which are considered particles, like protons and electrons, could turn out to have wave properties. Perhaps *everything* in the microscopic world exhibits wave-particle duality.

2.6.1 Electron Interference

What would this mean for an electron or a proton to have wave-like properties? The clear signature of waves is interference, so we could try to make electrons interfere like waves. Imagine an experiment where a beam of electrons is sent towards a double-slit apparatus, much like you did with a beam of light in lab. If the electrons are strictly particles with no wave character, we would expect to find two spots on the screen where the electrons are striking: one bright spot is the electrons that passed through the left slit and the other is the electrons that passed through the right slit. However, if the electrons are acting as waves, we should expect to see a full interference pattern on the screen, with a sequence of bright and dark spots.

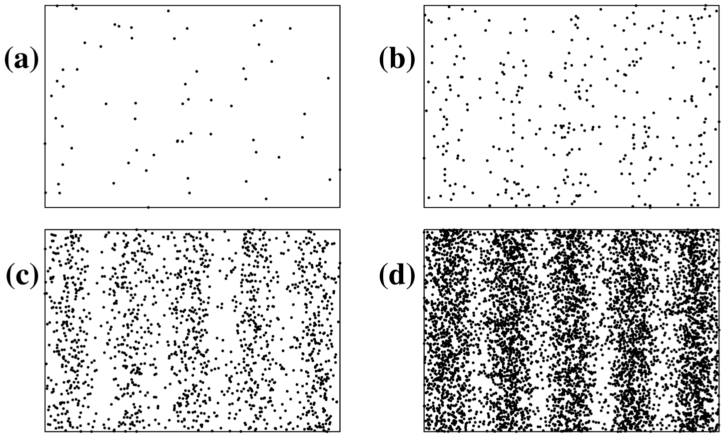


Figure 2.4: Electrons are sent through a double slit apparatus and then strike a screen. The dots represent where the electrons hit the screen. As the number of electron detections grows, the interference pattern becomes clear. (Simulated data.)

These experiments have been done. We show here a simulation of the data in Fig. 2.4. This figure demonstrates that once enough electrons have reached the screen, an interference pattern develops. Electrons can act as waves!

There is an extremely peculiar aspect to this experiment. First, note that the bright spots in the interference pattern are simply the regions where an electron hit more often. Equivalently, they are regions where an electron has a higher probability to hit. The electrons go through the double slit apparatus one at a time, and somehow they know where the higher and lower probability regions are. But the interference pattern is a property of both slits. For example, if the slit spacing is changed, the distance between interference bright spots is changed. This implies that somehow each single electron "experiences" both slits, since it passes through them knowing the probabilities of where to hit on the screen. This is very strange!

But very real. In fact, the wave-like property of electrons is the basis of a handy tool called the *electron microscope*.[6] This microscope can probe length scales significantly shorter than visible light to study, for example, cell organelles.

2.6.2 The de Broglie Relation

de Broglie went further than just proposing this wave-particle duality; he made a specific prediction for what the wavelengths should be for things we

---

[6] Bucknell has one. Some of you may have used it.

normally think of as particles. The de Broglie relation,

$$\lambda = \frac{h}{p} \tag{2.13}$$

says that the wavelength of a "particle" depends on its momentum and on Planck's constant.

Let us check that this relation works for photons: recall from the relativity unit of PHYS 211 that massless photons have an energy $E_{\text{ph}} = c|p_{\text{ph}}|$. The de Broglie relation says that the photon momentum and wavelength should be related by $p = h/\lambda$, so all together this gives

$$E_{\text{ph}} = c|p_{\text{ph}}| = c\frac{h}{\lambda} = hf. \tag{2.14}$$

Indeed, for photons Eq. (2.13) is equivalent to Einstein's photon energy relation.

So de Broglie's proposal was that $\lambda = h/p$ holds for all particles, not just for photons. Let's see how this works out for an electron microscope.

Example 3. Electron microscope.

You wish to use an electron microscope to resolve features of a cell organelle that are on the scale of 1 nm. To do this, you "shine" a beam of electrons on the sample. What is the minimum speed that these electrons could have?

Solution: Recall that to resolve features on a certain scale, we need a wavelength at least that small, so we will need $\lambda \leq 1\,\text{nm}$.

According to the de Broglie relation, this implies

$$p = \frac{h}{\lambda} \geq \frac{h}{1\,\text{nm}} \tag{2.15}$$

or

$$p \geq \frac{6.63 \times 10^{-34}\,\text{J·s}}{10^{-9}\,\text{m}} = 6.6 \times 10^{-25}\,\text{kg·m/s}. \tag{2.16}$$

Now that we have the minimum for the momentum, we can find the minimum speed simply by $p = mv$ (we are neglecting relativistic effects):

$$v = \frac{p}{m} \geq \frac{6.6 \times 10^{-25}\,\text{kg·m/s}}{9.11 \times 10^{-31}\,\text{kg}} = \boxed{7.3 \times 10^{5}\,\text{m/s.}} \tag{2.17}$$

So, we have seen that de Broglie argued (successfully) that particles act like waves. But *waves of what*? We will discuss this further in the next chapter.

Problems

1. Calculate:

   (a) the energy (in eV) of a 500 nm wavelength photon, and

   (b) the non-relativistic kinetic energy for a 500 nm electron.

2. A certain electron has the same wavelength as orange light, $\lambda \simeq 600\,\mathrm{nm}$. Calculate the speed of this electron.

3. The electron binding energy for a particular metal is 2.25 eV. Calculate the cutoff frequency for the photoelectric effect in this metal.

4. (a) Let's say that you want to probe the structure of a bacteriophage T4 virus.[7] One way to probe the virus is to look at it with electromagnetic waves. If one wants to "see" the T4 structure with a resolution of about 1 nm, what must be the wavelength of the EM wave used? What is the energy of a single photon with this wavelength?

   (b) Another way to probe the T4 is with an electron microscope. What must the wavelength and energy of an electron used by the microscope if it is to resolve the T4 virus down to the same resolution of $10^{-9}\,\mathrm{m}$?

5. Use the de Broglie relation to calculate the momentum of an X-ray photon of frequency $f = 1.0 \times 10^{18}\,\mathrm{Hz}$.

6. Photons of frequency $6.0 \times 10^{14}\,\mathrm{Hz}$ are directed towards a metal. As a result, electrons are ejected with kinetic energies up to 1.4 eV. Determine the binding energy for this metal.

7. A few years ago there was a flurry of attention given to the potential hazards of electromagnetic fields from overhead power lines. The concern is that the alternating current (AC) in these power lines was emitting radiation that could cause cancer.

   (a) The AC current in power lines alternates with a frequency of 60 Hz. Use this to determine the energy of photons emitted from the power lines (express your answer in eV).

   (b) The weakest molecular bonds have binding energies around 0.1 eV. Use this explain why the scientific community is highly skeptical of the claims of cancer dangers.

---

[7] These viruses — which attack common bacteria — are amazingly cool. They are like little lunar landers that land on the surface of a bacterium, after which they inject DNA into the bacterium, which then produces many more of the viruses before exploding. See, e.g., Kanamuru et al., Nature **415**, p. 553 (2002).

8. When a particular metal is illuminated with infrared radiation of wavelength 700 nm, electrons are emitted with kinetic energies that range up to 0.25 eV. Calculate the largest kinetic energy for ejected electrons if the same surface is illuminated with light of wavelength 400 nm.

9. A one-dimensional cavity of length L is filled with electromagnetic standing waves. Show that the frequency of the j^{th} longest wavelength mode is given by $f_j = cj/(2L)$.

10. (a) Given your results from Problem 9, calculate the energy of a single photon for the j^{th} longest wavelength mode in a one-dimensional (1D) cavity with length L.

(b) Considering that the Equipartition Theorem predicts an average total energy of $\frac{1}{2}k_BT$ for each mode, calculate the number of photons that you would expect to find (on average) for the j^{th} longest wavelength mode in a 1D cavity with length L.

(c) Based on your answer to (b), what would be the largest mode number j for which you would expect to find (on average) one or more photons in the cavity?

(d) In one or two sentences, explain why the photon nature of light resolves the problem of the UV catastrophe (i.e., the prediction of an infinite total energy).

11. As discussed in the reading, the Equipartition Theorem says that, classically, each electromagnetic wave mode should have an average energy of $\frac{1}{2}k_BT$. For simplicity, we'll assume a one-dimensional cavity with a length of 2.0 cm.

(a) Calculate the value of $\frac{1}{2}k_BT$ at room temperature of 22° C.

(b) Calculate the wavelength and frequency of the lowest frequency normal mode electromagnetic wave for this cavity.

(c) Calculate the energy of one photon of light with this wavelength.

(d) Calculate the number of photons that you would expect (on the average) for the lowest frequency mode in this cavity.

(e) How many photons would you expect (on average) for the second lowest frequency mode in this cavity?

(f) How many photons would you expect for the 10^{th} lowest frequency mode in this cavity? The 100^{th} lowest mode? The 1000^{th} mode? The 10000^{th} lowest mode?

(g) Do you think that the equipartition theorem holds for all of your answers in part (f)? Why or why not?

12. The electromagnetic field in a one-dimensional cavity is in thermal equilibrium, and the longest wavelength mode contains 4500 photons.

(a) Calculate the number of photons in the second-longest wavelength mode.

(b) Calculate the number of photons in the third-longest wavelength mode.

(c) Calculate $j_{\max}$, the largest mode number.

13. An X-ray photon of wavelength 60 nm ionizes a hydrogen atom with an electron initially in its ground state (with energy $-13.6\,\text{eV}$). Calculate the kinetic energy of the resulting free electron.

14. Wave-particle duality means that all the fundamental building blocks in the quantum microscopic world have both wave and particle properties. This is demonstrated in the electron double slit experiment shown in Fig. 2.4.

(a) Describe an aspect of the experiment that involves electrons acting as particles.

(b) Describe an aspect of the experiment that involves the same electrons acting as waves.

15. A He-Ne laser emits light of wavelength 633 nm. Calculate how many photons per second are emitted by a 5 mW He-Ne laser.

Chapter 3

Waves, Probabilities, and Uncertainty

3.1 Introduction

We concluded the last chapter with de Broglie's hypothesis that all matter exhibits wave-particle duality, not just light. De Broglie's proposal is radical. Arguing that all matter has wave-like properties turns out to have far-reaching implications whose answers fundamentally change the way we view the laws of the universe. One important question is the following: waves are spread out over regions of space; so, *where is the particle* if it is acting like a wave?

In this chapter, we will discuss the interpretation of de Broglie's waves in terms of *probability.* An implication of this probabilistic description of matter waves is the well-known *Heisenberg uncertainty principle*, which states that a particle cannot simultaneously have a precise position and momentum. We will show that the uncertainty principle ends up solving one of the failures of classical physics: it explains why atoms are stable. The uncertainty principle also limits our ability to measure and manipulate matter at small, sub-atomic scales.

We finish the chapter with a generalization of de Broglie's relation — Schrödinger's equation — that can be used to *calculate* wavefunctions for particles.

3.2 Quantum Waves

Let's consider a particle moving with a well-defined momentum p. According to de Broglie, this particle can also be thought of as a wave with a wavelength $\lambda = h/p$. We can represent this graphically as a simple sine

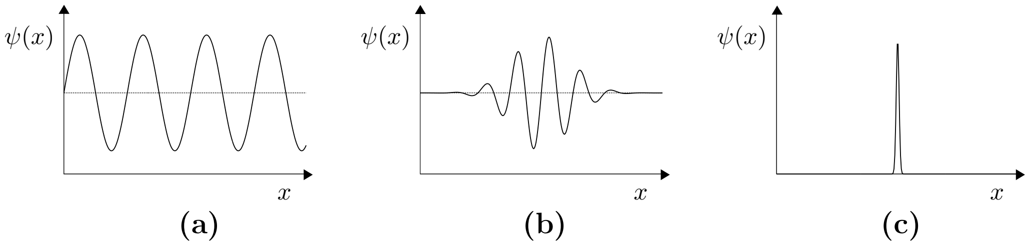


Figure 3.1: Wavefunctions $\psi(x)$ discussed in the text: (a) a pure sine wave, (b) a partially localized function, and (c) a localized blip.

wave, as shown in Fig. 3.1(a).[1] But a wave doesn't have to be a simple sine wave. The *wavefunction* $\psi(x)$ describing a particle's wave-like nature can take any shape. Figure 3.1(b) shows a wavepacket-shaped wavefunction, and Fig. 3.1(c) shows a pulse-like wavefunction. Later in this chapter, we will explain how the specific wavefunction can be determined for different physical situations.

But first: what exactly *is* the wave here? An answer to that question was provided in 1926 by Max Born who argued that the wavefunction contains information about the probability for finding the particle. Specifically, given a wavefunction $\psi(x)$ that describes the wave properties of a particle, Born argued that $\psi(x)$ represents *probability amplitude*, and taking the magnitude squared of the wavefunction produces the *probability density* $P(x)$ — probability per unit distance for a one-dimensional problem — for finding the particle:

$$P(x) = |\psi(x)|^2. \tag{3.1}$$

This tells us that according to quantum physics, a particle has a range of possible positions, described by a probability density. We don't know where the particle is located — there is a possibility that it could be found anywhere where the wavefunction $\psi(x)$ is non-zero. But it isn't just a statement that we don't *know* where the particle is; rather, *the particle doesn't HAVE a position* until we make a measurement and locate it.

The interpretation of matter waves as probability amplitudes has tremendous implications, which we will explore later in this chapter and in the next few chapters. But before we explore those implications, we need to develop the mathematics of probability densities.

3.2.1 Probability Densities

The concept of probability density can be tricky — it's not the same as probability! In one spatial dimension we write the probability density as

---

[1] Actually, a state with well-defined momentum has a complex wavefunction $\psi(x) = Ae^{2\pi ipx/h}$ whose real and imaginary parts are simple cosine and sine waves.

$P(x)$, since it can vary with the position x. To get probability from $P(x)$, you must multiply by a length interval since it is really *probability per unit length.* Specifically,

$$P(x)\,dx = \left(\begin{array}{l}\text{Probability of finding the particle within}\\ \text{an interval from } x \text{ to } x+dx.\end{array}\right). \tag{3.2}$$

If the probability density is uniform, then it is easy to find probability for any interval. To illustrate, suppose you are playing golf on a dark night (don't ask why) and after driving the ball you know it landed somewhere along the left edge of the fairway, between 200 to 250 meters from the tee.[2] You want to find it and learn about probability density at the same time.

The total probability of finding your ball between $= 200\,\text{m}$ and $x = 250\,\text{m}$ is 1 — it is definitely somewhere in that interval. Furthermore, the probability density is uniform. That is, the ball is equally likely to be found anywhere in the interval. Then the probability density $P(x)$ is given by

$$P(x) = \frac{\text{Total Probability}}{\text{Total Length of Interval}} \quad (\text{uniform } P(x) \text{ only}), \tag{3.3}$$

so in this case

$$P(x) = \frac{1}{(250\,\text{m} - 200\,\text{m})} = \frac{1}{50}\,\text{m}^{-1}. \tag{3.4}$$

If you limit your search to the region between $x_1 = 210\,\text{m}$ and $x_2 = 215\,\text{m}$, then your probability of finding the ball is

$$\begin{aligned}\text{Probability} &= (\text{Probability Density}) \times (\text{Length of Interval of Interest})\\ &= \frac{1}{50\,\text{m}}\,(215\,\text{m} - 210\,\text{m}) = \boxed{0.1 \text{ or } 10\%.}\end{aligned} \tag{3.5}$$

On the other hand, if the probability density is not uniform, you have to integrate:

$$\text{Probability of finding ball between } x_1 \text{ and } x_2 = \int_{x_1}^{x_2} P(x)\,dx. \tag{3.6}$$

This is the general relation you should use to determine probabilities for 1-D problems.

Finally, we note a useful graphical connection to probability. Since integrals have the nice interpretation of being area under a curve, lots of understanding of these concepts can be gained by plotting $P(x)$ vs. x, and simply calculating areas.

Example 1. Probability Density I.

A plot of a non-uniform probability density for finding your ball near a different hole is shown below, in Fig. 3.2.

---

[2] This is physics, so golf is played using the metric system.

Figure 3.2: Probability density for location of a golf ball.

(a) Show that the total probability of finding your ball is 1.

(b) Use the plot to find the probability of finding your ball between 150 and 200 m from the tee.

Solution:

(a) Since the total probability is the sum over the probabilities of being at all possible positions, we must integrate the probability density over all possible values of x, i.e., find the total area under the graph. The total net area is a triangle with a base of 100 m and a height of 0.02 m^{-1}, so

$$\text{Probability} = \text{Area} = \tfrac{1}{2} \times 100 \times 0.02 = 1. \tag{3.7}$$

.

(b) The probability is equal to $\int_{150}^{200} P(x)\,dx$, which is the area of the shaded trapezoid. The trapezoid can be considered as a rectangle (of base $b = 50$ and height $h = 0.01$) plus a triangle (of base $b = 50$ and height $h = 0.01$). Therefore the total probability is

$$\begin{aligned}\text{Probability} = \text{Area under curve} &= bh + \tfrac{1}{2}bh \\ &= 50(0.01) + \tfrac{1}{2}50(0.02 - 0.01) \\ &= \boxed{0.75 \text{ or } 75\%.}\end{aligned} \tag{3.8}$$

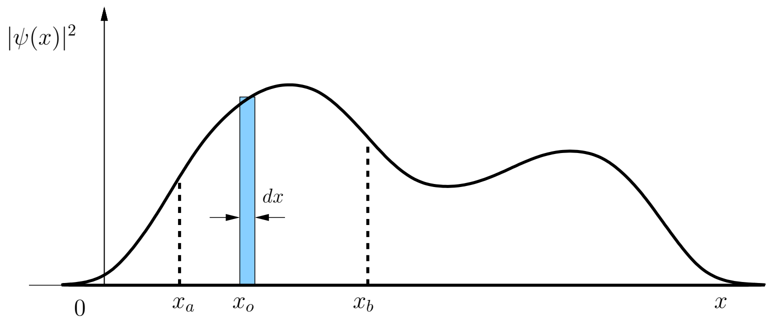


Figure 3.3: Determining the probability over a range of positions.

3.2.2 Probabilities and Wavefunctions

While a general wavefunction may be very complicated, the prescription for its interpretation is always the same: from $\psi(x)$, calculate $|\psi(x)|^2$, its magnitude squared. Then the probability density is given by $P(x) = |\psi(x)|^2$.

Figure 3.3 is the graph of a probability density for a particle in some quantum state. The quantity $P(x_0)\,dx$ represents the probability for finding the particle within a narrow interval from x_0 to x_0+dx, shown as the shaded area in the figure. If we would like to know the probability of finding the particle anywhere between the two positions x_a and x_b, then we need to *integrate* the probability density:

$$\left(\begin{array}{l}\text{Probability of finding the particle}\\ \text{in the region } [x_a, x_b]\end{array}\right) = \int_{x_a}^{x_b} P(x)\,dx = \int_{x_a}^{x_b} |\psi(x)|^2\,dx, \tag{3.9}$$

which is the area under the curve from x_a to x_b. Eq. (3.9) brings up an important property of wavefunctions and probability densities. What if the region we are interested in is the entire x-axis, that is, $[x_a, x_b] = (-\infty, +\infty)$? Another way of saying this is, 'What is the probability of finding the particle anywhere?' Certainly the answer to this question is 100% ! This implies that the wavefunction must be given such that

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\,dx = 1. \tag{3.10}$$

When a wavefunction satisfies this requirement, it is said to be *normalized*. Normalization is usually accomplished by multiplying the wavefunction by a suitable constant factor, as we will see in the following examples.

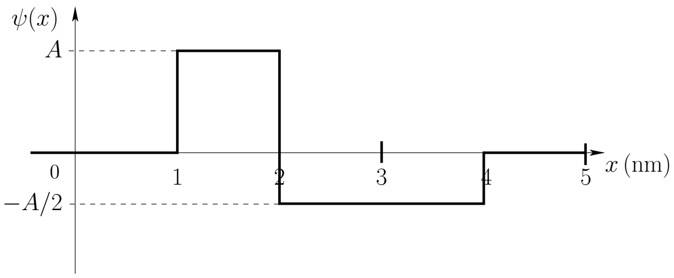


Figure 3.4: Wavefunction for Example 2.

Example 2. Probability of finding a particle.

Figure 3.4 shows the wavefunction for a certain particle, where A is a positive constant. (a) Determine a value for the constant A such that the wavefunction is properly normalized. (b) What is the probability of finding the particle in the region between $x = 2\,\text{nm}$ and $x = 4\,\text{nm}$?

Solution: (a) To determine probabilities we must first determine $|\psi(x)|^2$ for the given wavefunction. We do this by computing the value of $|\psi(x)|^2$ at every position on the graph which results in the graph of Fig. 3.5.

The total probability of finding the particle anywhere is the total area under the curve of Fig. 3.5:

$$\begin{aligned} \int_{-\infty}^{+\infty} |\psi(x)|^2\,dx &= A^2(2-1) + \left(-\frac{A}{2}\right)^2 (4-2) \\ &= A^2 + \frac{A^2}{2} = \frac{3}{2}A^2. \end{aligned} \tag{3.11}$$

For the wavefunction to be properly normalized, this area must equal

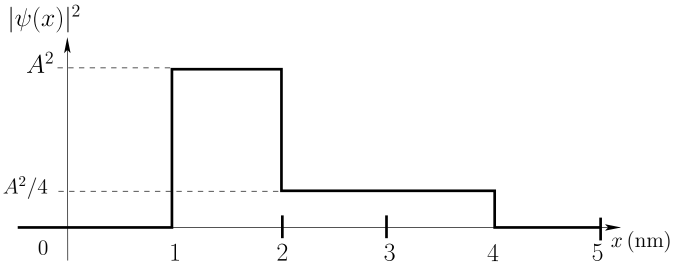


Figure 3.5: Probability density for Example 2.

one. Therefore, the value for the constant A must be

$$\frac{3}{2}A^2 = 1 \qquad \longrightarrow \qquad A = \sqrt{\frac{2}{3}}. \tag{3.12}$$

(b) Now that we know the value for the constant A that properly normalizes the wavefunction, the probability of finding the particle in the region between $x = 2\,\text{nm}$ and $x = 4\,\text{nm}$ is just the area under the curve of the probability density between $x = 2\,\text{nm}$ and $x = 4\,\text{nm}$:

$$\text{Prob}([2,4]) = \frac{A^2}{4}(4-2) = \frac{A^2}{2} = \frac{1}{3}. \tag{3.13}$$

3.3 Heisenberg's Uncertainty Principle

Okay, back to Fig. 3.1. We have already commented that a particle moving with a well-defined momentum p has a wavefunction $\psi(x)$ that is a pure sine wave, as in Fig. 3.1(a). But where is the particle when it is in this state? The answer: it doesn't *have* a well-defined position. There is a non-zero probability density for finding the particle anywhere where the wave is non-zero. But a pure sine wave with a definite wavelength extends all the way from $x = -\infty$ to $x = +\infty$. So a particle with a well-defined momentum could be found *anywhere*. Phrased another way, the spread in the particle's location is infinite. So, if the momentum of a particle is defined precisely, the particle's position is completely undetermined — it could be found anywhere. (And remember: it isn't just that we don't know where the particle is located, but rather that the particle simply doesn't *have* a location.)

But what if the particle's wavefunction looks like the one in Fig. 3.1(c)? In this case, $\psi(x) = 0$ everywhere except in a narrow region where the sharp blip is. So, its position is well-defined and predictable. It is theoretically possible to make an arbitrarily narrow pulse-like wavefunction, i.e., with a spread in position which is arbitrarily small. *But what is the wavelength — and therefore the momentum — of this particle?*

Formally, we use the symbol σ_x to denote the spread in a particle's position and σ_p to denote the spread in a particle's momentum.[3] For waves,

---

[3] Mathematically, σ_x is the same as the standard deviation

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}. \tag{3.14}$$

where the averages $\langle x\rangle$ and $\langle x^2\rangle$ are defined as

$$\langle x\rangle = \int x\,P(x)\,dx \qquad \langle x^2\rangle = \int x^2\,P(x)\,dx \tag{3.15}$$

with $P(x)$ being the probability density defined in the previous section.

there is an important theorem called *Fourier's theorem* that says that any function $f(x)$ — which might be describing the height of a water wave or the pressure of a sound wave as a function of position or a quantum wavefunction ψ — can be decomposed into a sum of sines and cosines.[4] We won't prove it here, but Fourier's theorem shows that a pulse-like wave with $\sigma_x = 0$ can be produced by adding sine waves with wavelengths ranging from 0 up to ∞. But since each wavelength corresponds to a different momentum, that means that a particle with a well-defined position (i.e., $\sigma_x = 0$) could have any momentum from infinity down to zero (i.e., $\sigma_p = \infty$).

So, a particle can have a well-defined momentum (but completely uncertain position) or a well-defined position (but completely uncertain momentum). There is a third possibility: Fig. 3.1(b) shows a wavepacket that is reasonably (but not perfectly) localized and one that can be decomposed into the addition of sine waves with a limited (but non-zero) spread of wavelengths. In this case, the particle is neither perfectly localized nor does it have a perfectly determined momentum, but neither σ_x nor σ_p are infinite either.

A strict application of Fourier's theorem to matter waves shows that there is a minimum total spread in the particle's position and momentum:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}, \tag{3.16}$$

where $\hbar \equiv h/2\pi$. This is known as the Heisenberg uncertainty relation. It is a profound result that caused Newton to roll over in his grave[5], because it says that we cannot simultaneously know a particle's position and momentum. A precise position and a precise momentum would mean both σ_x and σ_p are zero, but that is not allowed.

Recall when we solved for the motion of an object in classical mechanics, say a blow dart shot straight upward, we needed to know the initial position of the dart and the initial velocity. The Heisenberg uncertainty relation says that is not possible. The more precisely you are able to determine the position of some object, the less you will be able to know about its momentum.

Now, $\hbar$ is a really small constant. For distances and momenta on our macroscopic scale, Heisenberg's uncertainty places no practical limitations.

---

[4] Fourier's theorem is *really* cool and forms the basis for the theory of music among other things. The basilar membrane in your ear performs an operation very close to a Fourier transform, with different portions of the membrane responding to different sound frequencies. This process helps you to identify, say, the individual sounds produced by flutes, cellos and bass guitars that are all blended together during a concert. And Fourier analysis is used to determine the characteristic "fingerprint" of different musical instruments and, in fact, is used by designers of electronic music synthesizers to mimic the sounds of real instruments.

[5] Well, okay, maybe Newton didn't literally roll over in his grave, but this result is **very** different than anything envisioned by classical physics

But down at the atomic scale the Heisenberg uncertainty relation has a very big impact. And, of course, experiments have confirmed the uncertainty relation, so it is not just a proposal but it is part of reality.

Many people, when encountering the uncertainty relation for the first time, assume that this is a statement about the limits of our ability to do experiments. But that is not the case. It's a property of nature obeyed by all matter, even when we're not looking. As we shall see, the Heisenberg uncertainty relation is essentially the reason that atoms are stable.

3.4 The Stability of Atoms

Now we are ready to address the second great failure of classical physics. Maxwell's equations make a crystal clear prediction: accelerating charges send out energy in the form of EM waves. This is a big problem for the classical physics model of the atom. If the electrons orbit the nucleus like the planets orbit the sun, then they definitely have acceleration (equal to v^2/R), and so they are radiating away their energy.

But where would that energy come from? It comes from the potential energy of the electron's electric force interaction with the proton. To lower its potential energy, the electron must come closer to the proton, just like an object coming closer to the Sun lowers its gravitational potential energy.[6] So that is the classical solution: the electron must radiate away energy by coming ever nearer to the nucleus.

Working out the numbers involved for a hydrogen atom, we find that the electron should spiral in and essentially crash into the nucleus after a time of about 10^{-12} s. That is, classical mechanics and electromagnetism predict that atoms aren't stable and shouldn't last long enough for them (and us) to still be around. All the matter in our universe should be simply electrons sitting in the nucleus with the protons and neutrons.

Heisenberg's uncertainty principle resolves this dilemma. An electron cannot sit on top of the nucleus because that would be a very precise position and a very precise momentum (namely, at rest), which violates the uncertainty relation, so the best the electron can do is some compromise. It accepts the minimal spread in position and momentum that it can and then is stuck there. Since it cannot lower its energy further, it quits radiating and is stable.[7]

Put another way, whenever a particle is confined – i.e., if σ_x is less than ∞ – then there is a minimum spread $\sigma_p \neq 0$ in the momentum. Assume the product $\sigma_x \sigma_p$ in the Heisenberg uncertainty relation, Eq. (3.16), to be

---

[6] Actually, only half of the potential energy is radiated away. The other half goes into increased kinetic energy. But the total mechanical energy is still decreasing.

[7] Thank goodness!

as small as possible, namely, $\sigma_x \sigma_p \approx \hbar/2$. This gives us the inverse relation

$$\sigma_p \approx \frac{\hbar}{2\sigma_x}. \tag{3.17}$$

If $\sigma_p \neq 0$, there is a non-zero kinetic energy. The average momentum must again be zero by symmetry, $\langle p \rangle = 0$, so

$$\sigma_p^2 = \langle p^2 \rangle - \cancelto{0}{\langle p \rangle^2} = \langle p^2 \rangle. \tag{3.18}$$

The average kinetic energy can be related to the spread in momentum:

$$\langle K \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2m} \langle p^2 \rangle = \frac{\sigma_p^2}{2m}. \tag{3.19}$$

Conceptually, since the electron is confined, it is forced to have some spread in velocity or momentum. The larger this spread, the larger the resulting average kinetic energy.

So, what does this mean? If an electron in an atom were to spiral into the center, the spread in its position σ_x would get smaller and smaller. But smaller σ_x would result in a larger and larger σ_p (by the uncertainty principle) and a larger kinetic energy (larger than the classical increase in K from spiraling inward). For an electron in an atom, the reduction in its potential energy (by moving closer to the center of the atom) would be countered by an increase in its kinetic energy, which grows toward infinity if the electron spirals all the way in. There is an optimal radial distance where the mechanical energy $E = \langle K \rangle + \langle U \rangle$ reaches its minimum. So, the classical problem of an electron spiraling in doesn't apply, because eventually the energy *increases* if the electron gets closer and closer to the center of the atom.

So, the condition for the electron to spiral in all the way is gone. It can't keep radiating radiating EM waves, losing energy, and spiraling in to the nucleus of the atom. Heisenberg's principle indicates that the energy of the electron is not a minimum at the center of the atom but rather at a finite radius. So, an electron would need **more** energy to fall further into the center of the atom,

This is, of course, a simplification of the quantum explanation for the stability of atoms. As we'll see in the next chapter, the energy of an electron in an atom (and, in fact, the energy of any confined particle) is quantized, having only certain, discrete values.

In the meantime, an important consequence of Heisenberg's Uncertainty is the following:

A confined particle can never have zero kinetic energy.

We will say a lot more about this in the next chapter.

3.5 Schrödinger's Wave Equation

We have talked about the idea of a wavefunction $\psi(x)$ that describes the wave-nature of a particle, and we have introduced the idea that the wave represents probability amplitudes, with the probability density $P(x)$ for locating a particle being given by $P(x) = |\psi(x)|^2$. But how can we determine what that wavefunction is for a specific problem?

In 1926, Erwin Schrödinger proposed a differential equation whose solutions give the wavefunctions corresponding to matter waves. For a particle of mass m and total energy E, Schrödinger's equation written in one spatial dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x), \tag{3.20}$$

where $U(x)$ is the potential energy function of the particle. Schrödinger's equation is similar in form to the types of equations that describe other wave phenomena, such as sound waves and electromagnetic waves. The basic idea here is that, given a particle with mass m and a potential energy $U(x)$, Schrödinger's equation will allow us to determine the possible energies E and associated wavefunction solutions $\psi(x)$.

3.5.1 Testing Solutions to an Equation.

There are advanced techniques from the theory of differential equations that we could use to start finding solutions to Schrödinger's equation from scratch. But for this course, we use only the "guess and check" method, which is also a standard technique in solving differential equations. (In lab, you will solve Schrödinger's equation using a numerical iteration method.) The approach is quite simple: you guess a particular function, you put it into the equation and see if it works. If it works for all values of x, then you say that it is a solution. If it doesn't, then you say that it **isn't** a solution.

We will illustrate the "guess and check" approach with an example:

Example 3. Testing solutions to an equation.

Some equations have solutions which are functions, instead of just a number. Let's say you are trying to determine the function $f(x)$ that solves the following equation:

$$[f(x)]^2 - x^2 + 8x - 16 = 0. \tag{3.21}$$

You can probably solve this with algebra techniques, but let's pretend that you don't know how to solve it. Instead, we are going to test trial solutions.

(a) Test the solution $f(x) = ax + b$ to see if it satisfies Eq. (3.21). If it does, determine the values of the constants a and b for the solution.

(b) Test the solution $f(x) = ax$ to see if it satisfies Eq. (3.21). If it does, determine the value of the constant a for the solution.

(c) Test the solution $f(x) = ax^2 + b$ to see if it satisfies Eq. (3.21). If it does, determine the values of the constants a and b for the solution.

Solution: (a) Substituting $f(x) = ax + b$ into Eq. (3.21), we get

$$\begin{aligned}[f(x)]^2 - x^2 + 8x - 16 &= (ax+b)^2 - x^2 + 8x - 16 \\ &= a^2x^2 + 2abx + b^2 - x^2 + 8x - 16 \\ &= (a^2-1)x^2 + x(2ab+8) + (b^2-16)\end{aligned}$$

The question is whether this can all add up to zero for all values of x. The answer to that question is yes **only** if the constants, the x-terms and the x^2 terms all separately add to zero. That means that all of the terms in parentheses above need to add to zero separately. (You can't satisfy this by making $x = 0$, because then the function would only work for that particular value of x). So

$$\begin{aligned}b^2 - 16 = 0 &\rightarrow b = \pm 4 \\ 2ab + 8 = 0 \rightarrow 2ab = -8 &\rightarrow a = -4/b \\ a^2 - 1 = 0 &\rightarrow a = \pm 1\end{aligned}$$

All three of these relations work if $b = 4$ and $a = -1$ or if $b = -4$ and $a = 1$. So, we can say that $f(x) = ax + b$ *is* a solution to Eq. (3.21) if $b = 4$ and $a = -1$ or if $b = -4$ and $a = 1$, i.e., $f(x) = -x + 4$ and $f(x) = x - 4$ are both solutions.

(b) Now let's substitute $f(x) = ax$ into Eq. (3.21):

$$\begin{aligned}[f(x)]^2 - x^2 + 8x - 16 &= (ax)^2 - x^2 + 8x - 16 \\ &= a^2x^2 - x^2 + 8x - 16 \\ &= (a^2-1)x^2 + 8x - 16\end{aligned}$$

We could get rid of the x^2 term by making $a = \pm 1$, but we would still be left with $8x - 16 = 0$. The only way to make that work would be to say x must be equal to 2. But that means that the test solution $f(x) = ax$ (with $a = 1$) would only work at $x = 2$. If $f(x)$ only works at one particular value of x, then it isn't a solution of the equation. (Solutions must work for **every** value of x.)

(c) Now let's substitute $f(x) = ax^2 + b$ into Eq. (3.21):

$$\begin{aligned}[f(x)]^2 - x^2 + 8x - 16 &= (ax^2 + b)^2 - x^2 + 8x - 16 \\ &= a^2x^4 + 2abx^2 + b^2 - x^2 + 8x - 16 \\ &= a^2x^4 + (2ab^2 - 1)x^2 + (8)x + (b^2 - 16)\end{aligned}$$

This would work only if the coefficients of the x^4, x^2, x and constant terms were all zero. We can make the x^4 term disappear by making $a = 0$ and we can make the constant term disappear by making $b = \pm 4$, but that would leave non-zero coefficients for the x^2 and x terms. There is no combination of a and b that can make all four terms disappear, so there is no function of the form $f(x) = ax^2 + b$ that can satisfy Eq. (3.21).

The same approach works for **any** equation, including *differential equations* (i.e., equations that have derivatives in them).

3.5.2 Testing Solutions of Schrödinger's Equation

Okay, let's look at Schrödinger's equation now. We won't "solve" this equation (that requires mathematics beyond the scope of this course), but we will test possible solutions to see if they work. First, since Schrödinger's equation is a generalization of de Broglie's result, we should get the same result for a free particle (i.e., if $U(x) = 0$). De Broglie says that a free particle with a momentum p can be described as a sine wave with a wavelength $\lambda = h/p$. So, let's try a test solution $\psi_1(x) = A\sin(kx)$ and see if this satisfies Schrödinger's equation.

With $U(x) = 0$, Schrödinger's equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + 0 = E\,\psi_1(x). \tag{3.22}$$

The Schrödinger equation has the second derivative of the wavefunction in it, so let's calculate that derivative for our trial solution:

$$\frac{d\psi_1}{dx} = kA\cos(kx) \qquad \text{and} \qquad \frac{d^2\psi_1}{dx^2} = -k^2A\sin(kx).$$

Next, substitute the second derivative and the wavefunction into Eq. (3.22)

$$-\frac{\hbar^2}{2m}\left[-k^2A\sin(kx)\right] + 0 = E\left[A\sin(kx)\right] \tag{3.23}$$

and combine the $\sin(kx)$ terms

$$A\sin(kx)\left[\frac{\hbar^2k^2}{2m} - E\right] = 0. \tag{3.24}$$

This works for $A = 0$, but that isn't interesting because then $\psi(x)$ would just be 0. But $\psi_1(x)$ also works if the terms in brackets add up to zero, i.e., if

$$\frac{\hbar^2 k^2}{2m} - E = 0 \longrightarrow E = \frac{\hbar^2 k^2}{2m} \tag{3.25}$$

and

$$k = \pm\frac{\sqrt{2mE}}{\hbar}, \tag{3.26}$$

where k is the wavenumber $k = 2\pi/\lambda$. The mechanical energy E is equal to the kinetic energy K in this case (since $U = 0$ everywhere), so

$$\frac{2\pi}{\lambda} = \pm\frac{\sqrt{2m\left(\frac{1}{2}mv^2\right)}}{h/2\pi} = \frac{2\pi\sqrt{m^2v^2}}{h} = \frac{2\pi p}{h}. \tag{3.27}$$

We have dropped the ± since a negative wavelength doesn't have any physical relevance.) So, we can say that a test solution $\psi_1(x) = A\sin(kx)$ *does* satisfy Schrödinger's equation for a free particle ($U = 0$), assuming the wavenumber k has a value that corresponds to a wavelength $\lambda = h/p$, as expected by de Broglie's relation.

(Note that this process has not determined the value of A in our test solution. The trial function $\psi_1(x) = A\sin(kx)$ works for any value of A (assuming the correct k). The value of A needs to be determined by normalization conditions. We'll discuss this in the next chapter.)

For contrast, let's try a test solution that does *not* satisfy Schrödinger's equation.

Example 4. Trying another test solution for a free particle.

Test the trial solution $\psi_1(x) = Bx^2$ to see whether or not it satisfies Schrödinger's equation for a free particle.

Solution: With $U = 0$, Schrödinger's equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + 0 = E\,\psi_1(x).$$

Calculate the second derivative for our trial solution:

$$\frac{d\psi_1}{dx} = 2Bx \qquad \text{and} \qquad \frac{d^2\psi_1}{dx^2} = 2B. \tag{3.28}$$

Next, substitute the second derivative and the wavefunction into Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\left[2B\right] + 0 = E\left[Bx^2\right], \tag{3.29}$$

and cancel common factors and simplify, giving

$$EBx^2 + \frac{\hbar^2 B}{m} = 0. \tag{3.30}$$

In this equation there is no physically-meaningful choice of E and B that could make this true for **all** x,[8] so $\psi_1(x)$ is **not** a solution.

Finally, let's illustrate a harder problem, the quantum harmonic oscillator.

Example 5. Test solution for a particle in a harmonic oscillator potential.

Recall that the spring potential energy is given by $U(x) = \frac{1}{2}k_{\text{sp}}x^2 = \frac{1}{2}m\omega^2x^2$, where $\omega = \sqrt{k_{\text{sp}}/m}$. Inserting this into the Schrödinger equation, we find

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}k_{\text{sp}}x^2\,\psi(x) = E\,\psi(x). \tag{3.31}$$

Try the test function $\psi_2(x) = xe^{-ax^2}$ to see if this is a solution to Schrödinger's equation for a harmonic oscillator. If it is, determine the unknown constant a in the solution and the unknown energy E.

Solution: Again, we must calculate the second derivative of this function. We will leave the calculation of the second derivative for you to do (remember to be careful to correctly use the product rule and the chain rule). The final result for the second derivative is

$$\frac{d^2\psi_2(x)}{dx^2} = \left(4a^2x^3 - 6ax\right)e^{-ax^2}. \tag{3.32}$$

Now substitute this and $\psi_2(x)$ into Schrödinger's equation, Eq. (3.31):

$$-\frac{\hbar^2}{2m}\left(4a^2x^3 - 6ax\right)e^{-ax^2} + \frac{1}{2}k_{\text{sp}}x^2\left(x\,e^{-ax^2}\right) = E\left(x\,e^{-ax^2}\right). \tag{3.33}$$

Every term has the same exponential factor in it, so we cancel those to get

$$\frac{-2\hbar^2a^2}{m}x^3 + 3\frac{a\hbar^2}{m}x + \frac{1}{2}k_{\text{sp}}x^3 = Ex. \tag{3.34}$$

---

[8] $B = 0$ would work, but then you'd have $\psi_1(x) = 0$, which isn't a physically relevant solution, because $\psi_1(x) = 0$ means that there is no probability of finding the particle anywhere.

Now to finish, move Ex to the left hand side and collect common powers of x:

$$x^3\left(\frac{1}{2}k_{\mathrm{sp}} - \frac{2\hbar^2 a^2}{m}\right) + x\left(3\frac{a\hbar^2}{m} - E\right) = 0. \tag{3.35}$$

For this equation to hold for all x, we must make the coefficient of *each* power of x vanish separately. So both the coefficient of x^3 *and* the coefficient of x are set to zero:

$$\frac{1}{2}k_{\mathrm{sp}} - \frac{2\hbar^2 a^2}{m} = 0 \quad \text{and} \quad 3\frac{a\hbar^2}{m} - E = 0. \tag{3.36}$$

Solving these for a and E, we find

$$a = \frac{m\omega}{2\hbar} \quad \textbf{and} \quad E = 3\frac{a\hbar^2}{m} = 3\left(\frac{m\omega}{2\hbar}\right)\frac{\hbar^2}{m} = \frac{3}{2}\hbar\omega. \tag{3.37}$$

We see that $\psi_2(x) = xe^{-ax^2}$ *is* a solution to the Schrödinger equation if the constants a and E are as given in the previous equation. We see that Schrödinger's equation gives us the energy E for this state, and also determines the unknown constant a in our trial wavefunction, ψ_2. This wavefunction solution is actually the first excited state of the quantum oscillator: $\psi_2(x) = x\,e^{-(m\omega/2\hbar)x^2}$ with definite energy $E_2 = \frac{3}{2}\hbar\omega$. In one of the assigned problems at the end of the chapter, you will test the wavefunction solution corresponding to the ground state of the harmonic oscillator.

We'll say a lot more about quantization of energy states in the next chapter.

Problems

1. **Probabilities with dice.** This problem gets at the idea of a probability distribution. Take two of your dice, roll them, and record the sum of the spots showing on the top faces. (This should be a number between 2 and 12!) Repeat until you have recorded 20 results.

   (a) Collect your results in three "bins." How many trials do you find in the bin containing numbers 2, 3, 4, or 5? How many in the 6, 7, or 8 bin? In the 9, 10, 11, or 12 bin? Try to explain any patterns.

   (b) In problem session, combine your results with others to estimate the probabilities of getting a roll in each of the three bins.

   (c) Can you calculate theoretical probabilities? How do they compare with individual or class results?

2. **Classical probabilities for finding your car.** John, an aspiring physics student, works part-time parking cars at a downtown hotel. The lot is a long, underground tunnel, with all the cars parked in a single long row, 600 m long. When owners return for their cars, instead of telling them exactly where to find their cars, he describes the location in terms of probability and probability density.

   (a) Mr. Vanderbilt is told that his car "could be anywhere in the lot," which means that the probability density is constant. Calculate the value of this uniform probability density $P(x)$ for Mr. Vanderbilt to find his car a distance x from one end of the lot. (Answer in units of probability/m.)

   (b) Find the probability that Mr. Vanderbilt's car is in the first 100 m of the lot.

   (c) Mrs. Reeve is told that the probability density to find her car is a constant P_1 from $x = 0$ to $x = 200\,\text{m}$, and a second constant $P_2 = P_1/3$ in for $x = 200$ to $x = 600\,\text{m}$. Find the different constant probability densities P_1 for $0 < x < 200\,\text{m}$ and P_2 for $200\,\text{m} < x < 600\,\text{m}$.

   (d) Based on your results from part (c), find the probability that Mrs. Reeve's car is in the first 400 m of the lot.

3. The probability amplitude $\psi(x)$ for a certain particle to be at position x is

$$\psi(x) = \frac{\sqrt{x}}{a\sqrt{2}}, \qquad \text{for } 0 \leq x \leq a.$$

(a) Explain what the quantity $|\psi(x)|^2$ tells us about the particle.

(b) Calculate the probability that the particle is found between $x = 0$ and $x = a$.

(c) Calculate the probability that the particle is found anywhere else (i.e., **not** between $x = 0$ and $x = a$).

4. Determine the probability of finding a particle between $x = 1.0$ and $x = 2.0$ for the following wavefunction: $\psi(x) = 0.10x + 0.50$.

5. An electron in a hydrogen atom has a spread in position typically around $\sigma_x = 5 \times 10^{-11}$ m.

(a) Use the Heisenberg uncertainty relation to find a lower bound on the spread in momentum, σ_{p_x}, for the electron.

(b) Now find a lower bound on the spread in velocity, σ_{v_x}.

6. The Heisenberg uncertainty relation does not pose much of a limitation on our macroscopic scale. Consider a blow dart of mass 2.5 g. Let's imagine a measurement of the dart's position with a precision limited to 1 μm. Determine the lower bound on the spread in velocity imposed by the Heisenberg uncertainty relation.

7. Fabrication of nano-scale devices requires the ability to position atoms and molecules with very small spatial spread σ_x.

(a) Use Heisenberg's uncertainty principle to make a rough estimate of the precision by which a carbon atom (mass 2.00×10^{-26} kg and radius 70 pm) can be confined (i.e., determine the smallest σ_x) to keep its minimum kinetic energy below typical molecular binding energies of around 1 eV. Would you expect the uncertainty principle to be a problem if you wanted to arrange carbon atoms in a nanotech device with a precision of around one-hundredth of the radius of a carbon atom?

(b) Repeat the calculations in part (a) for a proton (mass 1.67×10^{-27} kg), also with $\langle K \rangle$ below 1 eV.

(c) Compare your result from (b) to the size of the nucleus of a carbon atom (2.7×10^{-15} m). What does this imply about a proton confined inside a carbon nucleus? I.e., what must be true for a proton to remain bound inside a carbon nucleus?

8. Back in 1975, Gordon Moore proposed that the number of transistors per area on integrated circuits roughly doubles every two years. This principle ("Moore's Law") has worked surprisingly well for 40 years now, with transistors introduced in 2012 as small as 22 nm and techniques are continually being developed to make them even smaller. But Moore's Law will eventually fail due to limitations imposed by the uncertainty principle.

(a) Use the uncertainty principle to calculate the smallest spread σ_x for an electron such that its minimum kinetic energy (due to uncertainty) is below the work function (binding energy) of silicon, which is 4.05 eV.

(b) Let's assume that the smallest possible transistor has an area 100 times the square of the σ_x that you calculated in part (a). Given the area (approximated as the square of 22 nm) for the best transistors from 2012, if Moore's Law continues to hold into the future, roughly what year will transistors reach this quantum limit?

9. Explain in a few sentences why classical physics (Newtonian mechanics and Electricity and Magnetism) predict that atoms with electrons orbiting around a nucleus are unstable and can't exist indefinitely.

10. Use the uncertainty principle to estimate the minimum $\langle K \rangle$ for

(a) an electron confined to a region of 50 pm (roughly the radius of a hydrogen atom);

(b) a DNA molecule (1.0×10^{-25} kg) confined to the nucleus of a cell 3.0 μm radius; and

(c) a 5.0 mg grain of sand in a pill box with width 2.0 cm.

11. Use the uncertainty principle to explain in 2 or 3 sentences

(a) why it is not possible for a confined particle to have zero kinetic energy; and

(b) why everyday-size objects (even something as small as a speck of dust) often seem to have zero kinetic energy, even when confined to a small region.

12. Given the equation $df/dx = 4.0 \sin(0.25x)$, test (by direct substitution) to determine if the following functions are solutions. If so, determine possible values of any constants.

(a) $f(x) = Bx^2$,

(b) $f(x) = B \sin(kx)$,

(c) $f(x) = B \cos(kx)$.

13. Given the equation $d^2 f/dx^2 = 5x + 6$, test (by direct substitution) to determine if the following functions are solutions. If so, determine possible values of any constants.

(a) $f(x) = Ax^3 + Bx^2 + Cx$;

(b) $f(x) = A \sin(kx)$.

14. Schrödinger equation for a classically allowed situation. Consider a particle of mass m in a region in which the potential energy is constant, i.e., $U(x) = U_0$, and assume that the total energy of the particle E is greater than the potential energy, i.e., $E > U_0$. (This is the case for classically allowed motion.) To determine the wave function we must find a function $\psi(x)$ that satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

In this problem you will try three "guesses" for $\psi(x)$ and see if they satisfy Schrödinger's equation. The three "guesses" are

- $\psi_1(x) = Ax^2$
- $\psi_2(x) = B\sin(kx)$
- $\psi_3(x) = Ce^{-\kappa x}$,

where A, B, C, k, and κ are undetermined real constants.

(a) Rearrange Schrödinger's equation so that the second derivative $d^2\psi/dx^2$ is alone on the left.

(b) Plug $\psi_1(x)$ into Schrödinger's equation and see if there is any choice for the constant A that will make $\psi_1(x)$ satisfy the equation for all values of x.

(c) Plug $\psi_2(x)$ into Schrödinger's equation and see if there is any choice for the constants B and k that will make $\psi_2(x)$ satisfy the equation for all values of x .

(d) Plug $\psi_3(x)$ into Schrödinger's equation and see if there is any choice for the constants C and κ that will make $\psi_3(x)$ satisfy the equation for all values of x.

(e) You should have found that $\psi_2(x)$ can be a solution for the proper choice of k. Determine the wavelength of the oscillations in terms of $\hbar$, m, E, and U_0. (i.e., solve for k and remember from the waves unit that $k = 2\pi/\lambda$.) Is your result consistent with that predicted from the de Broglie relationship? (Hint: $E - U_0$ is the kinetic energy $K = p^2/2m$. Re-write things in terms of the momentum and the answer should drop into your lap.)

15. Schrödinger equation for classically forbidden situation. Consider a particle of mass m in a region with a constant potential energy U_0, and assume that the total energy of the particle E is *less* than the potential energy, i.e., $E < U_0$. (This isn't possible for classical motion, but continue anyway.) To determine the wave function we must find a function $\psi(x)$ that satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$$

In this problem you will try three "guesses" for $\psi(x)$ and see if they satisfy Schrödinger's equation. The three "guesses" are

- $\psi_1(x) = Ax^2$
- $\psi_2(x) = B\sin(kx)$
- and $\psi_3(x) = Ce^{-\kappa x}$,

where A, B, C, k, and κ are undetermined real constants.

(a) Rearrange Schrödinger's equation so that the second derivative $d^2\psi/dx^2$ is alone on the left.

(b) Plug $\psi_1(x)$ into Schrödinger's equation and see if there is any choice for the constant A that will make $\psi_1(x)$ satisfy the equation for all values of x.

(c) Plug $\psi_2(x)$ into Schrödinger's equation and see if there is any choice for the constants B and k that will make $\psi_2(x)$ satisfy the equation for all values of x.

(d) Plug $\psi_3(x)$ into Schrödinger's equation and see if there is any choice for the constants C and κ that will make $\psi_3(x)$ satisfy the equation for all values of x.

16. A special case of the quantum harmonic oscillator is described by Schrödinger's equation of the form

$$-\frac{d^2\psi}{dx^2} + 4x^2\psi = E\psi.$$

Substitute the trial solution $\psi = Ae^{-x^2}$ and determine the value of the energy E.

17. The general case Schrödinger equation for the one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi.$$

Substitute the trial solution $\psi(x) = Ae^{-ax^2}$ (with $a > 0$) into this equation and determine the constants a and E. This is the ground state of the oscillator.

Chapter 4

Quantized Energies and Spectra

4.1 Introduction

We have seen that particles have wave-like properties, and that the de Broglie relation, connecting the properties of momentum and wavelength, has far-reaching consequences for the behavior of atoms and other microscopic systems. The main consequence is that the idea of a particle as a point-like object having a precise position at any given time has to be replaced with a probability density describing a distribution of positions where the particle is likely to be found.

In the previous chapter, we introduced the idea of a wavefunction $\psi(x)$ from which we can determine the probability density $P(x) = |\psi(x)|^2$. We also discussed Heisenberg's uncertainty principle, that states that there is a minimum combined spread in a particles position and momentum, a principle that also leads to a minimum kinetic energy for any confined particle. And at the end of the previous chapter, we introduced the Schrödinger equation which enables us to determine wavefunctions for particles in regions of given potential energy.

In this chapter, we show how Schrödinger's Equation predicts the *quantization* of energies for confined particles, i.e., only certain, well-defined energies are allowed.[1] Quantization of energies is a profound and distinctly quantum principle (i.e., not predicted at all from classical mechanics) which has extensive applications in modern technology. We will illustrate quantization of particle energies with a simple system — the *particle in a box* (also known as the *infinite square well potential*) — for which solutions of Schrödinger's Equation are readily obtained. We will discuss the properties of this system, which has recently led to the development of *quantum dots* for use in numerous applications.

---

[1] This is where the "quantum" in the name "quantum mechanics" comes from.

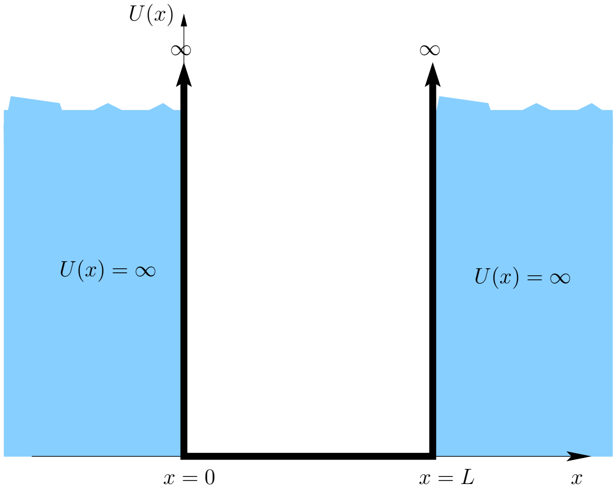


Figure 4.1: Graph of the infinite square well potential as a function of position.

We will also discuss how quantum systems absorb and emit light, a result that explains not only the colors that we see from many physical systems but which has also become a common tool for identifying the material constituents of a range of systems, from biological systems in microscopic studies up to stars and star-forming regions millions (and billions) of light years from the Earth.

4.2 Solutions for the Infinite Square Well

To demonstrate how we use Schrödinger's equation to derive wavefunction solutions, let's consider the simplest example of a "particle in a box," i.e., a particle trapped in one dimension (and unable to move in the other two dimensions) between two impenetrable walls located at $x = 0$ and $x = L$. Since the particle feels no force in the region between the walls, the potential energy in this region is constant, and therefore we will take it to be zero. Since the walls are impenetrable, the potential energy must become infinite at $x = 0$ and $x = L$. Therefore the graph of the potential energy $U(x)$ as a function of position is shown in Fig. 4.1. This configuration is usually referred to as the *infinite square well potential*. For this example we will consider solutions for which the energy of the particle E is positive. (Solutions for $E < 0$ for this potential actually do not exist due to the fact that the potential energy goes to infinity at the boundaries of the well.)

Since there is no chance that the particle can penetrate either of the barriers at $x = 0$ or $x = L$, the wavefunction $\psi(x)$ must be exactly zero in the regions $x \leq 0$ and $x > L$. Inside the well, where $U(x) = 0$, Schrödinger's equation can be written as

$$\frac{d^2\psi(x)}{dx^2} = -\left(\frac{2mE}{\hbar^2}\right)\psi(x). \tag{4.1}$$

Our job is to determine what mathematical function is consistent with Eq. (4.1), that is, can we find a function $\psi(x)$ whose second derivative is equal to *minus* a positive constant $(2mE/\hbar^2)$ times the original function? We already testing a solution to this equation in Section 3.5.2. In that section, we found that the wavefunction $\psi_1(x) = A\sin(kx)$ works if

$$E = \frac{\hbar^2 k^2}{2m}. \tag{4.2}$$

from which it follows that $k = \pm\sqrt{2mE}/\hbar$. This isn't the only possible solution — the wavefunction $\psi_1(x) = B\cos(kx)$ also satisfies Eq. (4.1). Since either of these functions satisfies Schrödinger's equation, we can write a general solution as a linear combination of these two functions:

$$\psi(x) = A\sin(kx) + B\cos(kx). \tag{4.3}$$

In order for this solution to work, we must adjust the values of the constants A, B, and k such that the wavefunction inside the well matches up with the value of the wavefunction at positions $x = 0$ and $x = L$, where we previously concluded that the value of the wavefunction should be zero. This process is called *matching the boundary conditions.* At $x = 0$, the wavefunction inside the well given by Eq. (4.3) becomes $\psi(0) = B$, so in order for the wavefunction to equal zero at the boundary $x = 0$, we must choose the value $B = 0$. Thus the interior wavefunction must be given simply as $\psi(x) = A\sin(kx)$.

Continuing, we now look at the boundary condition at $x = L$. Again, the wavefunction must be zero here, so this requires that

$$A\sin(kL) = 0. \tag{4.4}$$

From our knowledge of the sine function, we know that this function is zero for values such that $kL = 0, \pi, 2\pi, 3\pi, \cdots = n\pi$ for any integer n. Therefore this second boundary condition tells us that the possible values the constant k can assume are

$$k_n = \frac{n\pi}{L}. \qquad (n = 1, 2, 3, \ldots) \tag{4.5}$$

We have thrown out the $k = 0$ solution because that would correspond to a wavefunction $\psi(x) = 0$ which corresponds to a particle that has zero probability of being found anywhere, (i.e., the particle doesn't exist). Combining these values for k with Eq. (4.2), we find that the possible energies E_n that the particle can have inside the well are given by

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2} = n^2 \frac{h^2}{8mL^2} \qquad (n = 1, 2, 3, \ldots), \tag{4.6}$$

and the wavefunctions associated with each of these energies are

$$\psi_n(x) = A \sin\left(\frac{n\pi}{L}x\right). \tag{4.7}$$

Fig. 4.2 shows a graph of the allowed energies E_n for a particle in an infinite square well potential superimposed on the graph of potential energy $U(x)$ and Fig. 4.3 shows the wavefunctions $\psi_n(x)$ for the four lowest allowed energies.

A few general remarks are in order about these solutions.

- Notice that the lowest allowed energy for a particle is **not** zero! This coincides with our discussion in chapter 3 in which Heisenberg's uncertainty principle requires a minimum non-zero energy for a particle that is confined to a finite region of space.
- There are only certain values of the energy E for which there are well-behaved solutions to Schrödinger's equation. When a particle is trapped in a finite region of space, its energy cannot be any continuous value but rather can only be one of many discrete values of energy. In the language of quantum mechanics, the energy is *quantized.*
- The integer n is referred to as the *quantum number* of the particle. For this problem, specifying a value for n uniquely determines the state

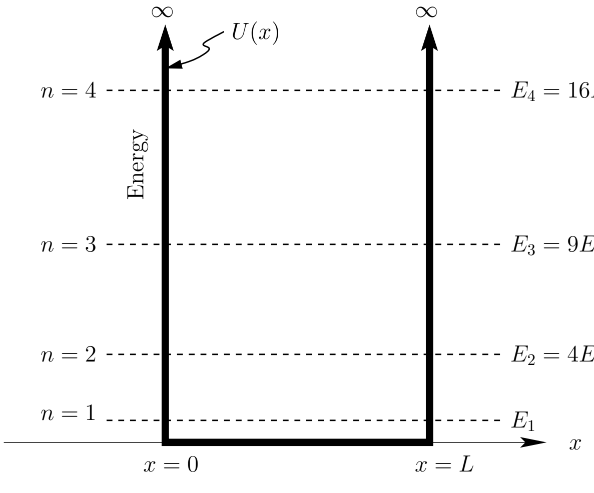


Figure 4.2: The allowed energies for a particle bound in an infinite square well potential. The energies are expressed in terms of the lowest (ground state) energy E_1.

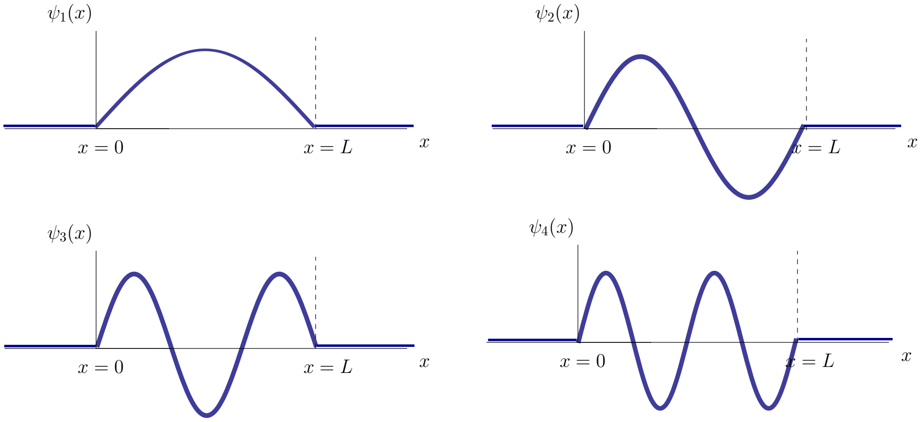


Figure 4.3: The wavefunction solutions for the infinite square well potential for the four lowest energy states.

of the particle. As we shall see in more complicated systems, more than one quantum number will often be needed to completely specify a state.

- The wavefunctions as given in Eq. (4.7) and Fig. 4.3 should look familiar to you. Aren't these the same functions we used when describing the standing wave patterns on a vibrating string fixed at both ends? Indeed they are! In fact, standing waves on a string are determined by a differential equation similar to Eq. (4.1). Standing waves on a string exhibit their own form of quantization – in this case, the properties that are quantized are the wavelengths and frequencies.

Example 1. Electron trapped in a 1-D nanotube.

Carbon atoms can be bonded into a cylindrical arrangement called a carbon nanotube. Carbon nanotubes can be fabricated with length-to-diameter ratios that are extremely large. These nanotubes have a broad range of applications including uses as conducting nano-wires and mechanical "scaffolding" for growing new bone cells.

A carbon nanotube of length L can be approximated as an infinite square well of width L. Consider a particular carbon nanotube of length $L = 10.0\,\mathrm{nm}$ and negligible diameter. Calculate the ground state energy for an electron trapped inside of this carbon nanotube.

Solution: Approximating the carbon nanotube as an infinite square well, the possible allowed energies for the electron are given by Eq. (4.6).

For the ground state energy, where $n = 1$, the energy is given as

$$E_1 = (1)^2 \frac{h^2}{8m_\mathrm{e}L^2}. \tag{4.8}$$

Because the length is given in nm, and eV's are convenient units for energy at the microscopic scale, it makes calculations easier if we multiply both the numerator and denominator by c^2, giving

$$E_1 = (1)^2 \frac{(hc)^2}{8(m_\mathrm{e}c^2)L^2}. \tag{4.9}$$

Then we can insert the values $hc = 1240\,\mathrm{eV{\cdot}nm}$, $L = 10\,\mathrm{nm}$, and $m_\mathrm{e}c^2 = 0.511\,\mathrm{MeV} = 511 \times 10^3\,\mathrm{eV}$, giving

$$\begin{aligned} E_1 &= (1)^2 \frac{(1240\,\mathrm{eV{\cdot}nm})^2}{8 \times 511 \times 10^3\,\mathrm{eV} \times (10\,\mathrm{nm})^2}. \\ &= 3.8\,\mathrm{meV} \end{aligned} \tag{4.10}$$

It is, of course, valid to use SI units for h, m_e, and L, but this entails calculations with large exponents and more unit conversions.

We can use the idea of normalization (see Sec. 3.2.2) to assign a value to the constant A in the wavefunction solutions of the infinite square well potential given in Eq. (4.7). Given that the wavefunction is zero outside of the well, the total probability of finding the particle within the well should be one. Therefore,

$$\int_0^L |\psi_2(x)|^2\,dx = \int_0^L A^2 \sin^2\left(\frac{n\pi}{L}x\right)\,dx = 1. \tag{4.11}$$

Using the table of integrals in Appendix A of your Wolfson text, we can evaluate the integral using

$$\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) = \frac{L}{2}. \tag{4.12}$$

Therefore

$$A^2\left(\frac{L}{2}\right) = 1 \qquad \text{and} \qquad A = \sqrt{\frac{2}{L}}. \tag{4.13}$$

Notice that this value of A is the same for all of the solutions, independent of n.

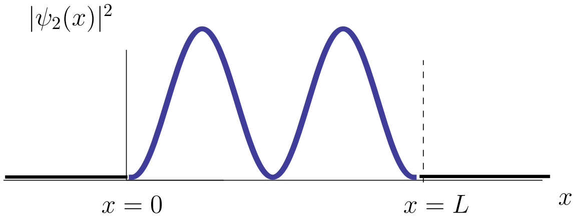


Figure 4.4: Probability density for a particle in the $n = 2$ state of an infinite square well.

Example 2. Probability of finding a particle in an infinite square well potential.

A particle of mass m is placed in an infinite square well of width L in a quantum state for which $n = 2$. (a) In the vicinity of what position(s) is the particle most likely to be found within the well? (b) What is the probability of finding the particle between positions $x = L/4$ and $x = L/2$?

Solution: (a) The wavefunction for this particle is shown as $\psi_2(x)$ in Fig. 4.3. Using Eq. (4.7), the probability density for this wavefunction is

$$|\psi_2(x)|^2 = \left(\sqrt{\frac{2}{L}}\right)^2 \sin^2\left(\frac{2\pi}{L}x\right), \tag{4.14}$$

as shown in Fig. 4.4. The positions near which the particle is most likely to be found are where the probability density is the greatest. By examining the graph of probability density, these positions are located at $x = L/4$ and $x = 3L/4$. **(b)** To calculate the probability of finding the particle between $x = L/4$ and $x = L/2$, we use Eq. (3.9):

$$\begin{aligned} \text{Prob}\left[\frac{L}{4}, \frac{L}{2}\right] &= \int_{L/4}^{L/2} |\psi_2(x)|^2 \, dx \\ &= \int_{L/4}^{L/2} \left(\frac{2}{L}\right) \sin^2\left(\frac{2\pi}{L}x\right) dx \\ &= \frac{2}{L}\left[\frac{x}{2} - \frac{L}{8\pi}\sin\left(\frac{4\pi}{L}x\right)\right]_{L/4}^{L/2} \\ &= \frac{1}{4} \end{aligned} \tag{4.15}$$

where the integral can be evaluated using techniques that you learned in your calculus class or the table of integrals in Appendix A of your Wolfson text.

Because of the symmetry of the probability density, the area under the curve of $|\psi|^2$ between $x = L/4$ and $x = L/2$ is one-quarter of the total area, therefore it makes sense that the probability is $\frac{1}{4}$.

4.3 Semi-infinite Square Well Potential and Quantum Tunneling

If we want to determine the wavefunction solutions for a particle confined in a region where the potential energy is some known function $U(x)$, then we perform a procedure similar to what we did for the example of the infinite square well potential. However, this time the potential is not zero (or a constant) but changes with position. Solving Schrödinger's equation for the case of a general potential energy function $U(x)$ can be very complicated and certainly this is beyond the scope of this course. In fact, there are very few potential energy functions for which the Schrödinger equation can be solved exactly.

However, we can give a slightly more complicated example of solving Schrödinger's equation where the potential energy has different constant values in two different regions of space, as shown in Fig. 4.5. In this case, the potential differs from the infinite square well potential in that the boundary at $x = L$ is now set to the finite value $U(x) = U_0$ for $x > L$ instead of being infinite. If we consider a solution for a total energy E which is less than the potential energy step U_0, then we see that Schrödinger's equation is different in the two regions $(0 < x < L)$ and $(x > L)$. Rearranging Schrödinger's equation, we obtain

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(U(x) - E\right)\psi(x). \tag{4.16}$$

We see that in the region $0 < x < L$, Schrödinger's equation reduces again to Eq. (4.1) so we should expect wavefunction solutions in that region to be similar to sinusoidal functions as given in Eq. (4.3). However, in the region $(x > L)$ the total energy is less than the potential energy $(E < U_0)$ and Schrödinger's equation becomes

$$\frac{d^2\psi(x)}{dx^2} = \frac{2m}{\hbar^2}\left(U_0 - E\right)\psi(x) \qquad \text{for } x > L. \tag{4.17}$$

As we did before, we must determine a mathematical function $\psi(x)$ that is consistent with Eq. (4.17), remembering that the quantity $[U_0 - E]$ is a positive quantity. This time the second derivative of $\psi(x)$ is equal to a *positive* number times the same function. In this case, sinusoidal functions will not work, but again, if we think back to our calculus course, we recall that

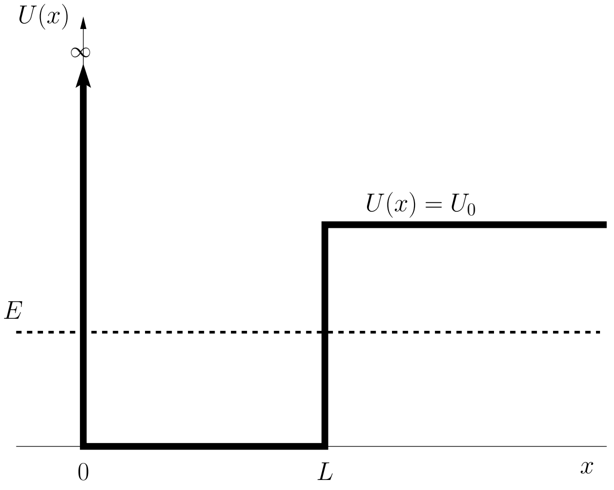


Figure 4.5: The semi-infinite square well potential energy function.

another set of functions we know is the right choice here — the exponential functions $Ce^{+\kappa x}$ and $De^{-\kappa x}$, where C, D, and κ are constants, with $\kappa > 0$.

Example 3. Wavefunction solutions for $E < U$.

Given the potential shown in Fig. 4.5, show that the function $Ce^{\kappa x}$ is a solution to Schrödinger's equation (4.17) and determine what the constant κ must be for this to be a solution.

Solution: To see if $\psi(x) = Ce^{+\kappa x}$ is a solution, we first evaluate the left hand side of Eq. (4.17) by calculating the second derivative of $\psi(x)$:

$$\begin{aligned}\frac{d\psi(x)}{dx} &= \kappa Ce^{\kappa x} \\ \frac{d^2\psi(x)}{dx^2} &= \kappa^2 Ce^{\kappa x}. \qquad (4.18)\end{aligned}$$

Inserting this into Schrödinger's equation (4.17) we find

$$\kappa^2 Ce^{\kappa x} = \frac{2m}{\hbar^2}\left(U_0 - E\right) Ce^{\kappa x}. \qquad (4.19)$$

In order for $\psi(x) = Ce^{+\kappa x}$ to be a solution, the left side of this equation must equal the right side, and therefore

$$\kappa^2 = \frac{2m}{\hbar^2}\left(U_0 - E\right). \qquad (4.20)$$

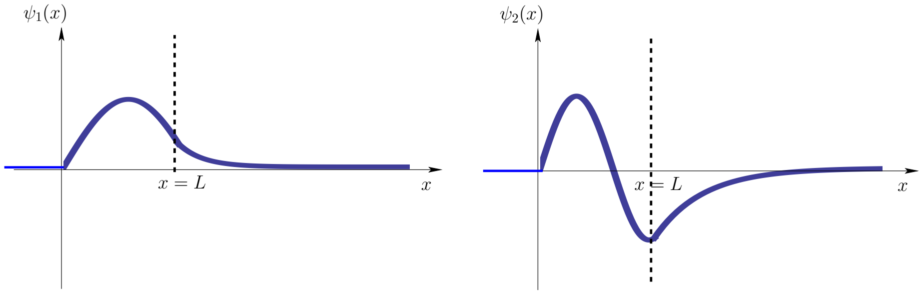


Figure 4.6: The lowest two energy states of a particle in a semi-infinite potential well.

Example 4. Complete wavefunction solution for the semi-infinite square well.

The complete wavefunction for the semi-infinite square well is made by combining the three solutions to Schrödinger's equation in the three regions, $(x < 0)$, $(0 < x < L)$, and $(x > L)$:

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ A\sin(kx) + B\cos(kx) & \text{for } 0 < x < L \\ Ce^{+\kappa x} + De^{-\kappa x} & \text{for } x > L \end{cases} . \tag{4.21}$$

As we did in the previous section, we can further simplify this solution by matching this solution to what the wavefunction should be at $x = 0$ and as $x \to \infty$. At $x = 0$, the wavefunction should be zero so that it matches $\psi(x) = 0$ for $(x < 0)$. This requires that the constant $B = 0$ so the function $\cos(kx)$ does not appear. As $x \to \infty$, the function $e^{+\kappa x}$ would become infinite and is therefore a not well-behaved solution. So we choose $C = 0$ to eliminate that function from our solution and the final solution becomes

$$\psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ A\sin(kx) & \text{for } 0 < x < Lr \\ De^{-\kappa x} & \text{for } x > L \end{cases} . \tag{4.22}$$

The wavefunctions in the two regions given in Eq. (4.22) must match at position $x = L$ such the the wavefunction and its first derivative make a smooth transition from the region $x < L$ to the region $x > L$. The mathematics is beyond the scope of this course, but there are only certain values of k, and hence E, for which the wavefunctions make a smooth transition. Graphs of the wavefunctions for the two lowest energy states $\psi_1(x)$ and $\psi_2(x)$ are shown in Fig. 4.6.

In some of the problems at the end of the chapter you will use an Excel spreadsheet to determine solutions to the semi-infinite square well using a numerical method.

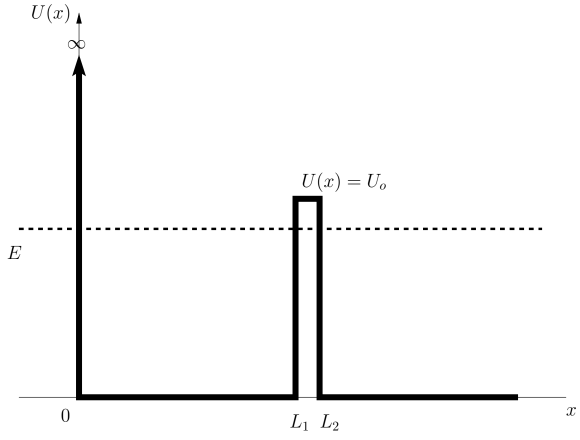


Figure 4.7: For this potential, a particle with energy E in the region $0 < x < L_1$ would remain trapped forever, according to classical physics. But quantum theory predicts that the particle can "tunnel" through the thin barrier and escape.

An examination of Fig. 4.6 leaves us the following important conclusions:

- Wavefunctions in the region where $E > U$ are sinusoidal functions similar to those in the infinite square well potential. The wavelength depends on the difference $|E - U|$.
- Wavefunctions in the region where $E < U$ are exponential functions.
- In the case of the semi-infinite well, the length of the well L is not exactly an integer or half-integer multiple of the wavelength as in the infinite square well (see Fig. 4.3).

Note also that in the region where $E < U$ the wavefunction $\psi(x)$ is *not zero*. This might not seem at first glance to be all that surprising until you consider that a particle's kinetic energy $K = E - U$, and the region where $E < U$ is a classically forbidden region (i.e., a particle can't ever be found in that region, classically) since classical mechanics forbids a particle ever from having a negative kinetic energy. But even though classical mechanics says that a particle can never enter a region where $E < U$, we can see from this example that quantum theory predicts a non-zero probability for a particle to be found in this region.

But what if there is a potential well such as the one shown in Fig. 4.7? If a particle confined in the region $0 < x < L_1$ has an energy $E < U_0$, then according to classical theory, that particle will remain trapped between $x = 0$ and $x = L$ forever, since it can never pass through the classically-forbidden region $L_1 < x < L_2$. But we just saw in our previous example that Schrödinger's Equation predicts a non-zero probability of a particle being found in a classically-forbidden region, so presumably, there is a non-zero probability that the particle in Fig. 4.7 can be found in the region $L_1 < x < L_2$. But if it can make it into that region, then there is a non-zero probability that the particle can be found **outside** the confined region, i.e. for $x > L_2$! In other words, a classically-trapped particle can "tunnel" out of the trap in quantum mechanics.

We won't calculate the probabilities for tunneling. But examination of Fig. 4.6 indicates that the probabilities tail off the wider the classically-forbidden region is. So, the probabilities can become quite small unless the barrier is very narrow and unless the energy E is close to the barrier height U_0. Despite these limitations, quantum mechanical tunneling can play a very important role in various processes, and has already found its way into some useful applications:

- Scanning tunneling microscopes (STMs) have been developed that use tunneling to produce images of surfaces that can resolve individual atoms.
- Josephson junctions and tunneling diodes have become tools of modern electronics.
- Tunneling is being proposed as a new mechanism for developing transistors, which could improve the performance of integrated circuits in the future.

4.4 Energy Bands

Quantized energies are found for particles confined in any system, not just the infinite or finite square well potentials. Schrödinger's Equation can be solved to determine the allowed energies for particles in any potential energy function. (In fact, determining allowed energies for electrons in atoms is one fundamental aspect of physical chemistry.) Each element (and molecule) has its own pattern of energy levels. As we'll see later this chapter, this has implications for the light that is absorbed and emitted by different elements and molecules.

But in addition to the case of a single electron trapped in a single potential well, there are many systems composed of multiple electrons trapped in a repeating pattern of potential wells. Consider a chunk of aluminum in a crystalline pattern, for example. The nuclei of the aluminum atoms produce a potential energy function that has repeating energy wells that mobile electrons experience.

An interesting thing happens to the quantized energy levels when there is more than one side-by-side potential well. For simplicity, we will consider the case of side-by-side, 1D, finite square-well potentials. Figure 4.8) shows

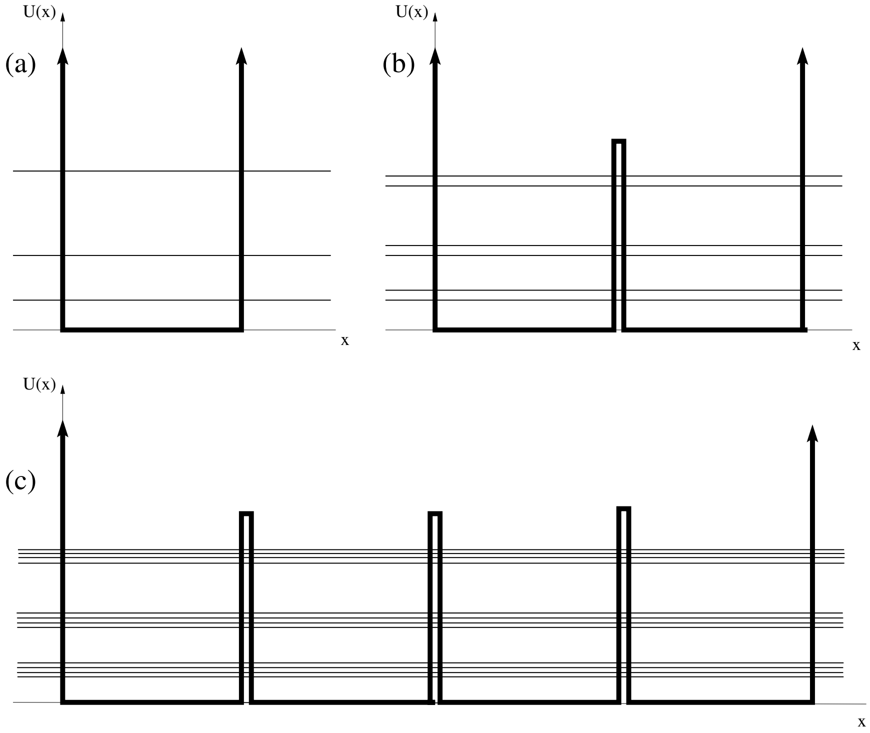


Figure 4.8: Allowed energy levels for (a) infinite square well potential; (b) double, side-by-side potential wells, separated by a thin, finite barriers; and (c) four side-by-side potential wells.

a representation of what happens to the energy levels when more and more square-well potentials are added to the system. Each energy level in the single finite-well potential (Fig. 4.8a) splits into two nearby levels for the double-well potential (Fig. 4.8b). If there are four side-by-side potential wells (Fig. 4.8c), there are four energy levels clustered around each value. As the number of side-by-side wells increases, the number of energy levels in each "band" increases accordingly.

In a real solid (such as aluminum), there are many, many, many side-by-side potential wells, so many that it becomes difficult to distinguish the individual energy levels within each band (Fig. 4.9). They are still there — in the figure, it may look like a continuous band of allowed energies, but there are still a finite number of allowed energy levels (although a **very** large number of finite values) in each energy band. So, even though the energies appear continuous within each band, they are still quantized.

In Fig. 4.9, we show only two bands of allowed energies, although there are typically more. The valence band is the highest energy band that is typically filled with electrons.[2] The conduction band always has available energy

---

[2] As we will see in Chapter 6, the Pauli Exclusion Principle argues that two electrons

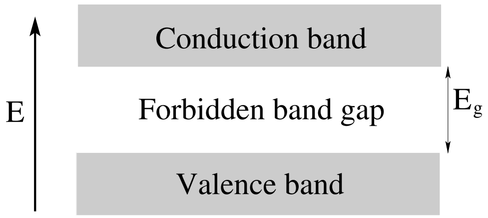
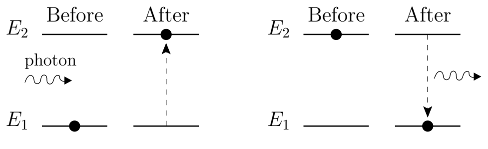


Figure 4.9: Valence and conduction bands for a typical solid composed of many atoms. Each of these bands contains a large (but finite) number of discrete energy levels.



(a) absorption

(b) emission

Figure 4.10: (a) Absorption of a single photon, causing an electron in the material to jump up to a higher (unoccupied) energy state. (b) Emission of a photon by an electron that drops from a higher to lower energy state.

levels that are unoccupied. In Chapter 7, we'll discuss how the band structure discussed here is important in understanding electrical conductivity for solids, and how the quantum properties of these bands have been used to develop some of the most important building blocks of modern electronics.

4.5 Absorption, Emission, and Spectroscopy

Okay, we are now ready to address some questions that you might have asked when you were a kid: "Why is a tomato red?" or "How do glow-in-the-dark shirts work?" or "Why is it that black-light illumination (which looks kinda deep violet in color) cause certain pigments to glow orange" or "Why do certain laundry detergents make my clothes look so unnaturally bright white?" These questions all fall under the category of the topic of photophysics or photochemistry, i.e., the physics and chemistry of how light interacts with matter. In addition to providing answers to these questions, this subject also relates to the important topic of *spectroscopy* which is a critical diagnostic technique that spans all fields of science and engineering.

There are a few different ways in which light (and any electromagnetic radiation) can interact with matter, but two important processes are absorp-

---

cannot occupy the same state, so there is a finite number of electrons that can have energies in any band in a solid.

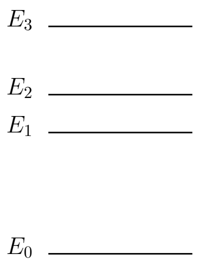


Figure 4.11: Example of an atom or molecule with four quantized energy levels.

tion and emission of individual photons of light. Figure 4.10 shows these two processes. If a photon strikes a material, it can be absorbed by an electron, resulting in an increase in the energy of that electron from an initial (lower) value E_1 to a higher value E_2, as shown in Fig. 4.10a. This process must conserve energy, which means that the energy gained by the electron must be the same as the energy of the absorbed photon:

$$E_{\text{ph}} = \Delta E = |E_{\text{final}} - E_{\text{initial}}| \tag{4.23}$$

Note that the upper energy level state must initially be unoccupied for this process to work. As we'll see in Chapter 6, two electrons cannot occupy the same quantum state.

The process of emission is the same thing in reverse (Fig. 4.10b). An electron in a higher energy state can drop to a lower-energy state (if initially unoccupied), releasing a single photon with an energy given by Eq. (4.23).

These simple quantum principles now enable us to resolve the third of the great failures of classical physics — the observation in the early 1900s that when energy is pumped into atoms, they radiate EM waves only at certain distinct wavelengths; i.e., the problem of discrete atomic spectral lines discussed in Chapter 2. The explanation is actually quite simple: each element or molecule has its own discrete, quantized energy level structure (e.g., Fig. 4.11). An electron in an atom can make transitions between the energy levels. Any transition from a higher to a lower energy level will result in the emission of a photon. But since there are only certain, well-defined, quantized energy levels for the electron, then there are only certain well-defined *differences* ΔE allowed for the transitions between these levels. And since the photons emitted by these transitions have energy $E_{\text{ph}} = |\Delta E|$, that means that the emitted photons have only certain, well-defined energies. And since the energy of the photon is related to the frequency and wavelength of the electromagnetic radiation via $E_{\text{ph}} = hf = hc/\lambda$, that means that the light emitted by atoms has only certain, well-defined, discrete wavelengths.

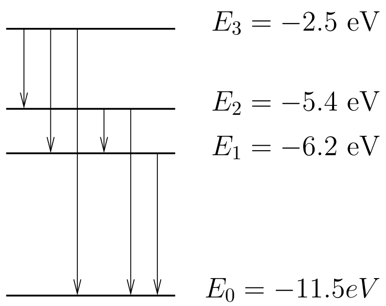


Figure 4.12: Possible transitions that result in emission of EM radiation for the 4-level system of Example 5.

Example 5. Light emitted from a molecule with four energy levels.

For a molecule with the four-level system in Fig. 4.11, assume that $E_0 = -11.5\,\text{eV}$, $E_1 = -6.2\,\text{eV}$, $E_2 = -5.4\,\text{eV}$, and $E_3 = -2.5\,\text{eV}$. (a) Determine how many different wavelengths of EM radiation can be emitted from this molecule, and calculate (b) the largest and (c) the smallest wavelengths of EM radiation that can be emitted from this molecule.

Solution: (a) There are several possible transitions that an electron can make that will emit a photon in this system. An electron can drop from energy level 3 down to level 0, 1 or 2; it can drop from energy level 2 down to level 0 or 1; and it can drop from energy level 1 down to level 0. Altogether, there are 6 possible downward energy transitions, so there are 6 possible wavelengths of light that can be emitted.

For parts (b) and (c), for any emitted photon, the energy $E_{\text{ph}} = |\Delta E|$ and since $E_{\text{ph}} = hc/\lambda$, it follows that $hc/\lambda = |\Delta E|$ and $\lambda = hc/\Delta E$. So, the largest and smallest wavelengths correspond to the smallest and largest energy differences ΔE. (b) The smallest energy difference ΔE given the allowed electron energies is the energy difference between levels 2 and 1. So, the largest emitted wavelength is:

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\text{eV}\cdot\text{nm}}{|-5.4\,\text{eV} - (-6.2\,\text{eV})|} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.8\,\text{eV}} = 1550\,\text{nm}, \tag{4.24}$$

where we have used the result $hc = 1240\,\text{eV}\cdot\text{nm}$, which is convenient for spectra problems where the energy levels and differences are expressed in electron volts and the wavelengths of emitted and absorbed light are expressed in nanometers. We could have converted everything into SI units and used typical values for h and c, but that is more time-consuming.

Note that the longest wavelength emitted is a value beyond the visible spectrum (this is in the infrared spectrum), so the human eye wouldn't be able to see this emitted light. (c) For the shortest wavelength for emitted EM radiation, we want the largest ΔE possible for these four energy levels, which corresponds to a transition from energy level 3 down to energy level 0.

$$\lambda = \frac{hc}{\Delta E} = \frac{1240\,\mathrm{eV{\cdot}nm}}{|-2.5\,\mathrm{eV} - (-11.5\,\mathrm{eV})|} = \frac{1240\,\mathrm{eV{\cdot}nm}}{9.0\,\mathrm{eV}} = 138\,\mathrm{nm}$$

This wavelength is also outside the visible spectrum (it's ultraviolet, in this case).

Everything in the previous example works the other way for absorption. But instead of an electron dropping to a lower energy level and emitting a photon, a photon is absorbed and the electron jumps up to a higher energy level. But this only works if the incoming photon has the right energy. From a practical perspective, if you shine white light through a gas of a particle element or molecule, all the light will pass through **except** for light with particular wavelengths and energies corresponding to the energy differences ΔE for the sample.

Since each element and each molecule has its own distinctive energy-level structure, the spectrum of EM radiation emitted is unique to each, as is the spectrum of radiation absorbed. This "fingerprint" is often used to identify materials in systems as diverse as a microscopic sample of DNA[3] up to astronomical-scale objects such as stars and planets; large, nebular clouds that are collapsing to form new stars and planetary systems; and accretion disks of gas spiraling into supermassive black holes at the centers of "active" galaxies.[4]

So, why is a tomato red? Answer: because it contains a pigment called lycopene that has electronic energy levels with an energy difference corresponding to the energy of red photons. When an electron in an excited state in lycopene drops to a lower level, it emits red photons. (Lycopene is actually a fluorescent pigment; fluorescence is discussed in the next section.)

---

[3] Spectroscopy is often used in the biological sciences. We have had many biology majors in PHYS 212 in the past who have commented that they see these techniques used quite often in the biology labs at Bucknell.

[4] Spectroscopy is the most important tool for identifying materials in distant astronomical objects. It's not as though we can go collect a sample of gas from a nebula 100,000 light years away. The light that we see from these objects is the only data that we have in most cases.

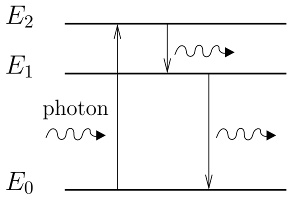


Figure 4.13: Fluorescence: Energy-level diagram for a three-level atom, showing absorption of a photon with large energy and emission of two photons with smaller energy.

Finally, absorption and emission also work for solids with band structures similar to that in Fig. 4.9. There are many, closely-spaced energy levels within the bands, allowing for EM radiation with very large wavelengths to be emitted and absorbed. But it is also possible for electrons to make transitions between the bands, resulting in larger energy differences and smaller wavelengths for emitted or absorbed light.

Example 6. Absorption by a material with a band gap.

A light detector uses a material with a filled valence band (all available energy states occupied by an electron) and an empty conduction band separated by a band gap energy of 3.2 eV. Calculate the largest wavelength of light most effectively absorbed by this detector.

Solution: The energy of the photon $E_{\text{ph}} = |\Delta E|$ and since $E_{\text{ph}} = hc/\lambda$, it follows that $hc/\lambda = |\Delta E|$. The largest wavelength corresponds to the smallest energy difference ΔE. Since the valence band is filled, the nearest unoccupied energy level is at the bottom of the conduction band, so $\Delta E_{\text{min}} = 3.2\,\text{eV}$. So $\lambda_{\text{max}} = hc/\Delta E = (1240\,\text{eV}\cdot\text{nm})/(3.2\,\text{eV}) = 387.5\,\text{nm}$.

4.6 Fluorescence and Phosphorescence

Many materials can absorb one wavelength of EM radiation and emit EM radiation with a different wavelength. Uranine dye (sodium fluorescein), for example, readily absorbs near-UV radiation, but emits green light. The principle — which is called "fluorescence" — is quite simple, and is very common in materials with more than two possible electronic energy levels.

Figure 4.13 shows the basic idea for fluorescence. A large energy photon (e.g., corresponding to ultraviolet light) is absorbed by a low energy electron, which jumps up more than one energy level. The electron can later spontaneously drop back down to a lower energy level, but in addition to dropping back down directly to the original state, it can also drop back down to an intermediate state and then later drop to the original state. For a three-level system like in Fig. 4.13, there are three different energies possible for the emitted light: $E_{\rm ph} = \Delta E_{31}$, ΔE_{32}, or ΔE_{21}.

So, why does black light illumination cause certain pigments to glow at different colors? Assume that the pigment has an energy level structure similar to that in Fig. 4.13. An incoming photon from the black light — which is really near-UV radiation with a "color" just beyond the visible (although there is usually a little violet light that comes out of a black light as well) — has enough energy to cause an electron in the ground ($n = 0$) state to become excited to the $n = 2$ state. After an undetermined amount of time, the electron spontaneously drops back down to a lower energy level. If it drops directly back to the $n = 0$ level, it emits another near-UV photon, which you don't really see. But if it drops first to the $n = 1$ level and then later to the $n = 0$ level, it emits two lower-energy photons, one of which (and maybe both) has energy in the visible part of the EM spectrum. (For many fluorescent pigments, one photon will be in the visible range, e.g., orange or green, and the other will be an infrared photon which you can't see.) So, that's why some materials fluoresce at a different color than they are illuminated with.

Fluorescent pigments can be found in many applications. Of course, there are many toys with fluorescent pigments, including Crayola^{TM} crayons (with fluorescent colors such as *screamin' green*, *atomic tangerine*, and *unmellow yellow*). Fluorescent dyes are also used in laundry detergents to make clothes appear brighter (that's why white clothes glow when illuminated by black light).

But fluorescence is also a *very* important tool used in scientific studies. Fluorescent "tags" are often used in biological studies. The basic idea is that if you can attach a fluorescent marker to a particular molecule in a biological system, then if you illuminate your sample with near-UV radiation, only the tagged regions will fluoresce in the visible, enabling you to see (usually under a microscope) the particular molecules in question. In recent years, biologists have developed a fluorescent protein called "green-fluorescent protein" (or GFP for short) that can be attached to various molecules in a living cell. Better yet, techniques have been developed to create targeted mutations in microorganisms that result in the GFP tag being an *inherited* property of the organism (i.e., they are born fluorescing.)

Fluorescence is also being used to revolutionize surgery with a new (as of 2017) technique referred to as "fluorescence image guided surgery." The idea is quite simple: a fluorescent dye that preferentially bonds to cancerous cells is injected into a patient. During surgery, the region is illuminated with high-frequency light (possibly near-UV), and the fluorescent dye attached to the cancerous tumor glows, enabling the surgeon to cut away the minimum amount of cancerous tissue without removing healthy tissue. This is a technique that figures to significantly improve survival rates for cancer patients during the next decade.

Another very new technology is that of the "quantum dot," which is a micro-engineered particle-in-a-box. We talk a lot about the particle-in-a-box problem because it is easy to solve, *but these things are actually being made!!* And the nice thing about a quantum dot is that the energy level structure can be altered simply by changing the size of the quantum dot (i.e., the width of the square-well potential).

A quantum dot is a a semiconductor that is manufactured to be **so** small (about 10 – 50 atoms in diameter) that its charge carriers experience effects of quantum confinement, including quantized energy levels. The result is that the filled valence band and empty conduction band for the semiconductor are replaced by a smaller number of discrete energy levels, similar to those of the particle-in-a-box. But since the electrons are confined (due to the small size of the dot), the minimum conduction band energy is shifted upward (away from the band) gap by a particle-in-box energy. Similarly, the maximum valence band energy is shifted downward. The result is the lowest energy E_{ph} of an emitted photon for a quantum dot is

$$E_{\text{ph}} = E_g + \eta \, \frac{h^2}{8 m_{\text{e}} R^2}, \tag{4.25}$$

where m_{e} is the electron mass and η is a factor that includes all of the material-specific semiconductor effects.[5] Since there are higher, open energy levels an electron in the quantum dot can absorb smaller wavelength (higher energy) photons and then fluoresce by emitting larger wavelength (smaller energy) photons as it drops back down in energy.

Example 7. Quantum dot fluorescence.

A quantum dot made from a CdTe semiconductor (with band gap energy 1.38 eV and $\eta = 20.5$) is fabricated with a radius 4.22 nm. Calculate the longest wavelength emitted by this quantum dot.

Solution: The longest wavelength corresponds to the smallest energy (Eq. (4.25)). Since, $E_{\text{ph}} = hc/\lambda$, and using the trick employed in Example 1 of multiplying the numerator and denominator of the second

---

[5] The constant η includes deviations from the standard particle-in-box energies due to the fact that this box isn't empty; rather, the electron is moving within a semiconductor.

term by c^2, it follows that:

$$\begin{aligned}\frac{hc}{\lambda} &= E_g + \eta\,\frac{h^2}{8m_{\mathrm{e}}R^2} \\ &= E_g + \eta\,\frac{(hc)^2}{8(m_{\mathrm{e}}c^2)R^2} \\ &= 1.38\,\mathrm{eV} + 20.5\,\frac{(1240\,\mathrm{eV{\cdot}nm})^2}{8\times(511\times10^3\,\mathrm{eV})\times(4.22\,\mathrm{nm})^2} \\ &= 1.81\,\mathrm{eV}. \qquad (4.26)\end{aligned}$$

Rearranging this gives

$$\begin{aligned}\lambda &= \frac{hc}{1.81\,\mathrm{eV}} \\ &= \frac{1240\,\mathrm{eV{\cdot}nm}}{1.81\,\mathrm{eV}} \\ &= 684\,\mathrm{nm}. \qquad (4.27)\end{aligned}$$

One final question in this chapter: *when will an excited electron drop back down in energy, emitting a photon?* The answer: you can never know. There is nothing in quantum theory that enables you to predict the precise moment when the emission process will happen. **However**, it is possible to calculate a probability that an emission process will occur within a certain time range, similar to how we can calculate a probability for a particle to be found in a certain region, and it is also possible to define a *half-life* for emission processes; i.e., the duration over which (on the average) half of the atoms in an excited state will have decayed back to the ground state, emitting photons.

The half-life for spontaneous emission can vary dramatically from one material to another. Usually, the half-life is a very short time; e.g., nanoseconds and even picoseconds. But there are some materials that have unnaturally long half-lives. In these cases, the material will continue to emit light. Some composites made from $SrAl_2O_4$ have been made with half-lives of several minutes, which means that an excited sample of this material will emit light for quite a long time after it is excited.

This is the principle (called *phosphorescence*) behind “glow-in-the-dark” pigments and materials. You “charge up” the material by exposing it to visible or near-UV light, promoting electrons in the pigment to excited energy levels. These electrons then drop back to the lower energy levels, but over the course of several minutes. During that time, the pigment appears to “glow,” even if there are no lights around.

Problems

1. (a) Draw a sketch of the wavefunction $\psi(x)$ for the 3^{rd} lowest energy state for an electron trapped in a one-dimensional, infinite square well potential (i.e., the "particle in a box").

(b) Interpreting your drawing from part (a) as a standing wave pattern, determine the wavelength λ (in terms of the width L of the box) of this standing wave.

(c) Substitute the wavefunction $\psi(x) = \sqrt{2/L}\,\sin(2\pi x/\lambda)$ with your value of λ from (b) into Schrödinger's Equation for the infinite square well potential. Verify that this is a solution for the region $0 < x < L$, and determine the energy of the state. Compare your result with what you get when using the formula from Eq. (4.6).

2. Find the energies (in eV) of an electron in (a) the ground state; and (b) the first excited state of a 1.0 nm one-dimensional infinite square well potential.

3. A particle is in the ground state of an infinite square well between $x = 0$ and $x = L$. What is the probability of finding the particle in the region between $x = 0$ and $x = L/3$? (**NOTE:** You may want to make use of the Table of Integrals in Appendix A of your Wolfson text for this problem!)

4. Sketch the probability density for the $n = 2$ state (first excited state) of an infinite square well extending from $x = 0$ to $x = L$. In the vicinity of what position(s) is the particle most likely to be found?

5. Let's apply what we have learned about infinite square wells to a macromolecule confined to a biological cell. Consider a protein of mass 250,000 u (where $1\,\text{u} = 1.661 \times 10^{-27}$ kg) confined to a 10 μm-diameter cell. Treating this as a particle in a one-dimensional square well, find the energy difference between the ground state and the first excited state. Given that biochemical reactions typically involve energies on the order of 1 eV, what so you conclude about the role of quantization in these reactions?

6. The potential energy for the one-dimensional *finite* square well is shown in Fig. 4.14, with dotted lines representing the energies of the ground state and the first excited state.

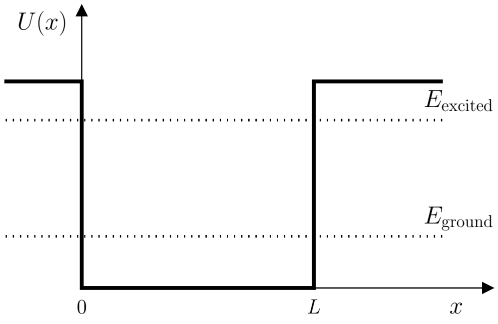


Figure 4.14: Plot of $U(x)$ versus x, for Problem 6.

(a) Using general principles developed in Examples 3 and 4, sketch the ground state wavefunction versus position, including regions in which $E < U$.

(b) Compared to the ground state wavefunction of the infinite square well with the same width, is the wavelength in the classically-allowed region longer or shorter? Is the energy larger or smaller?

(c) Repeat a) and b) for the first excited state.

(d) If you did (a) - (c) correctly, you should notice that the wavefunction isn't zero for $x < 0$ or for $x > L$, so there is a non-zero probability that the particle could be located in these regions. Why is this a violation of classical physics? (Hint: what can you say about the kinetic energy and speed of the particle when it is in one of these regions?)

7. The wavefunction solution to the semi-infinite square well was given in Eq. (4.22) for the two regions $0 < x < L$ and $x > L$:

$$\psi(x) = \begin{cases} A\sin(kx) & \text{for } 0 < x < L \\ De^{-\kappa x} & \text{for } x > L \end{cases} \tag{4.28}$$

In order for the wavefunction solution to be continuous and smooth across the boundary at $x = L$, the value of the wavefunction and its first derivative for the two solutions must match up at the boundary at $x = L$.

(a) Using the solutions given in Eq. (4.22), write an equation that states that the two wavefunction solutions are equal at the boundary $x =$

L. (Your equation should only contain the symbols k, κ, A, D and numerical constants.)

(b) Using the solutions given in Eq. (4.22), write an equation that states that the *d*erivatives of the two wavefunction solutions are equal at the boundary $x = L$. (Your equation should only contain the symbols k, κ, A, D and numerical constants.)

(c) Using the Schrödinger equation, it can be shown that the quantities k and κ are related to the energy E of the particle according to

$$k^2 = \frac{2mE}{\hbar^2} \qquad \text{and} \qquad \kappa^2 = \frac{2m\,(U_0 - E)}{\hbar^2} \tag{4.29}$$

where m is the mass of the particle and U_0 is the height of the potential well. Given these relations and the results of parts (a) and (b), do you think that the particle can have *any* value of energy E in the well?

8. **Semi-infinite square-well potential.** Download the Excel worksheet `semi-finite.xls` from either the *Handouts* page or from the Calendar page. This sheet shows the calculations for determining the wavefunctions for a potential well that is infinite at $x = 0$ but of finite magnitude on the right side of the well (which is at $x = 5$ in this problem). You'll see two graphs: the top one shows the semi-infinite potential well (in purple) along with a non-normalized plot of the calculated wavefunction so you can see it along with the potential. The bottom graph shows the normalized wavefunction, corresponding to the second-to-last column in the worksheet.

When you bring up the worksheet, the energy will be set for the value for the ground state. Some questions:

(a) Sketch or print out (just the first page!) the wavefunctions that are displayed for the ground state along with at least two of the excited states. To display the 1^{st} and 2^{nd} excited states, type in 0.64282 and 1.4144 respectively in the framed box for energy.

(b) What happens if you type in an energy that **isn't** one of the well-defined energies for the problem? Try it out, and comment on what happens. Had we not told you what the allowed energies were, how might you figure them out? (You'll be doing this in lab later this semester.)

9. For the potential energy function $U(x)$ and total mechanical energy E in Fig. 4.7:

(a) Use classical physics to argue that if a particle starts out in the region $0 < x < L_1$, it will remain there forever.

(b) Use quantum arguments (specifically, refer to wavefunction) to argue that a particle that starts in the region $0 < x < L_1$ can escape (tunnel). Can you predict *the precise moment* when the particle will escape? (Hint: no.)

10. Given the solution to Schrödinger's Equation for an electron in the semi-infinite square well potential (see Examples 3 and 4). Assume that $U_0 = 12\,\text{eV}$ and the electron is in a state with an energy $E = 10\,\text{eV}$.

(a) Calculate the value of the constant κ in the exponent $e^{-\kappa x}$ for the wavefunction in the classically-forbidden region.

(b) Calculate the distance into the classically-forbidden region beyond $x = L$ where the probability density is a factor of 2 smaller than that at $x = L$ (i.e., calculate the value of d such that $|\psi(L + d)|^2/|\psi(L)|^2 = e^{-2\kappa d} = 0.5$.

(c) Now, let's assume that the particle is a ball with mass 0.5 kg, and assume that U_0 and E have everyday values of, say, 100 J and 50 J, respectively. Repeat the calculations from parts (a) and (b).

(d) Now consider the potential energy function in Fig. 4.7. What do you think is needed to get a reasonable (not ridiculously small) probability for the particle to tunnel through the barrier? Why do you think we never experience quantum tunneling for everyday objects?

11. Consider a proton ($m_p = 1.67 \times 10^{-27}\,\text{kg}$) confined in an infinite potential well of size $L = 10^{-14}\,\text{m}$. Calculate the energy of a photon emitted when the proton makes a transition from the $n = 4$ state to the $n = 3$ state.

12. Electrons in an ensemble of 10 nm wide one-dimensional infinite square-well potentials are all initially in the $n = 4$ state. Find the wavelengths of all possible photons emitted as electrons make transitions to the ground state.

13. Calculate the radius of a quantum dot made from CdSe (band gap energy 1.74 eV and $\eta = 16.5$) so that the largest emitted wavelength will be 629 nm.

14. (a) Ozone (O_3) in the atmosphere absorbs ultraviolet radiation that can damage the skin and cause cancer. It does this via a photochemistry process $O_3 + \gamma \rightarrow O_2 + O$, where γ is a photon. If the dissociation energy of ozone is 3.94 eV, calculate the maximum wavelength of UV radiation that is absorbed by this photochemical process.

(b) Diatomic oxygen (O_2) also absorbs UV radiation with smaller wavelengths via the process $O_2+\gamma \rightarrow O+O$. Given a dissociation energy of 5.17 eV for an O_2 molecule, calculate the maximum wavelength of UV radiation that is absorbed by this process.

15. Titanium dioxide is one of many different possible ingredients used in sunblock lotions. It has a band gap energy of 3.2 eV. Calculate the maximum wavelength of radiation that you would expect TiO_2 to absorb via an absorption process that promotes electrons from the valence to the conduction band. Is your answer consistent with the use of TiO_2 as an ingredient in sunblock lotions?

16. Carbon dioxide has first and second excited vibrational states which are 0.083 eV and 0.166 eV above the ground state. Calculate the wavelengths of electromagnetic radiation that you would expect to be most readily absorbed by transitions between these vibrational energy levels in CO_2.

The wavelengths that you will find are in the infrared range of 700 nm to 1 mm. This is one of the reasons why carbon dioxide contributes to global warming as IR radiation is one of the ways in which the Earth radiates heat, assuming it isn't absorbed in the atmosphere. (Note, though, that this is a simplification, as there are rotational energy levels in CO_2 as well that aren't included in this problem.)

17. Class 3 (neutral phenol) Green Fluorescence Protein (GFP) used in cell and molecular biology studies has an excitation wavelength of 399 nm (near-UV) and an emission wavelength of 511 nm (green, duh).

(a) Determine a 3-level energy diagram (with ground state energy zero) that is consistent with these data. (There are two answers that are consistent with the given data.)

(b) Based on your diagram, you might expect to see two different colors of emitted light. Why do you see only green light emitted from a sample tagged (or genetically mutated) with Class 3 GFP?

18. The spec sheets for Lumidot^{TM} quantum dots from Sigma-Aldrich scientific supply company specifies the following core sizes (radius of the quantum dots) and wavelengths for their various dots.

Radius (nm)	Wavelength (nm)
3.0	510
3.3	530
3.4	560
4.0	590
5.2	610
6.3	640

Use this data to verify the $1/R^2$ dependence of Eq. (4.25) and to find the band gap energy. (Hint: Calculate E_{ph} for each of these wavelengths, and then plot E_{ph} versus $1/R^2$; a computer can make the calculations, plotting, and fitting easier.)

Chapter 5

Quantum States and Spin

5.1 Introduction

In the previous chapters, we have emphasized the use of the wavefunction $\psi(x)$ to describe the quantum state of a particle. This wavefunction formalism is mainly focused on answering the question "Where is the particle?" by virtue of the fact that the square of the magnitude of the wavefunction, $|\psi(x)|^2$, represents the probability density for finding the particle near the location x. This wavefunction approach is useful in describing how electrons arrange themselves in an individual atom as well as describing how electrons behave in a conductor, among other things. On the other hand, there are certain systems for which the wavefunction approach is neither appropriate nor meaningful, as for example the quantum description of photon states and, as we shall see later in this chapter, the quantum description of spin. Therefore in this chapter we will develop a more abstract mathematical structure in which to describe quantum systems that is more widely applicable and also reveals more of the non-intuitive quantum behavior. We will use this new mathematical formalism to describe a property of elementary particles called spin angular momentum, or simply *spin*, and how the spin of the particle interacts with a magnetic field. This description has important implications for understanding nuclear magnetic resonance (NMR) which is the basis for magnetic resonance imaging (MRI).

5.2 State Representation for Quantum Systems

In order to proceed further with our discussion of quantum mechanics, we will develop a mathematical framework in which to discuss the more general behavior of quantum systems that will allow us to perform calculations and make predictions that can be compared with experiment. As stated above, the wavefunction formalism gives a description of quantum behavior in which measurement of the position of the particle is of prime importance. A more encompassing mathematical structure for describing quantum behavior was devised by the English theoretical physicist P.A.M. Dirac in 1930 that is capable of describing a wide variety of phenomena, including photons and spin.

Fundamental to this new description is the concept of the quantum *state*. Mathematically, the state of a particle is represented by the symbol $|\phi\rangle$ called a "state" or "ket vector." The quantity ϕ written inside the symbol $|\ \rangle$ usually gives some meaningful description which uniquely describes the state in which the particle is at a certain time. For example, if a particle was in a state of constant momentum p, moving uniformly through space not subject to any forces, then we could write $|p\rangle$ for its state vector. These state vectors satisfy a set of mathematical rules similar to those for ordinary vectors in 3-dimensional space (like $\vec{A}$, $\vec{B}$, etc.), hence the reason for referring to them as state *vectors*.

As an example, we have seen that for a one-dimensional infinite potential square well, a particle placed in this well can have a discrete set of possible energy values (E_1, E_2, E_3, …, E_n, …). For each of these values of energy we can define a state vector ($|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, …, $|E_n\rangle$, …) that represents the system in a state with that definite value of energy. So for a particle that is placed in the infinite well in a state $|E_3\rangle$, if we were to measure the particle's energy we would definitely get the value E_3 as a result of that measurement. The collection of state vectors $\{|E_1\rangle, |E_2\rangle, |E_3\rangle, \ldots\}$ for all possible states is said to be a *basis*, (in this case an *energy* basis). This set of state vectors is analogous to the set of unit vectors $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ that we used with ordinary vectors in three-dimensions. Recall that we could write any 3-dimensional vector $\vec{A}$ as a linear combination of these unit vectors

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k} \tag{5.1}$$

where A_x, A_y, and A_z are the x-, y-, and z-components of $\vec{A}$. In an analogous fashion we can write an expression for an arbitrary state of a quantum system as a linear combination of the energy basis states

$$|\phi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle + \cdots\ , \tag{5.2}$$

where the coefficients c_1, c_2, …, are complex numbers. Complex numbers are defined as $c = a + ib$, where a and b are real numbers and i is the imaginary number $i = \sqrt{-1}$. The *real* part of the complex number is a, and b is the *imaginary* part. It may seem strange to introduce complex numbers into our quantum states, but it is actually essential. And fortunately, as we will see, anything physical that can be measured, such as a position, an energy, a probability, etc. will turn out to be real and not have an imaginary component.

In the mathematical structure of quantum mechanics, these coefficients c_1, c_2, … have a special significance when making predictions about the results of measurements on a system. If we were to measure the energy of a particle in a state given by Eq. (5.2), then the probability that we will measure a value E_3, for instance, is given by $|c_3|^2$. That is, the coefficients tell us something about the probability of finding a particle in a certain specific state of the measured quantity, such as energy in this case. It is for this reason that the coefficients c_j are referred to as *probability amplitudes*.

It is important to note that for complex numbers such as the coefficients c_j, the magnitude squared $|c_j|^2$ does not just mean "square the coefficient". Rather you should follow this three-step process:

1. Write the *complex conjugate* of the coefficient, c_j^*, by replacing every occurrence of "i" with "$-i$" in the wavefunction. For example if $c_1 = 0.2 - 0.5i$, then $c_1^* = 0.2 + 0.5i$.
2. The magnitude squared is defined as $|c_j|^2 = c_j\, c_j^*$, so instead of squaring c_j, multiply c_j times c_j^*.
3. Replace any occurrences of "i^2" with "-1" and simplify. Your result should be real (no left-over "i"s) and greater than or equal to zero, as required for a probability.

Example 1. Superposition of states.

Suppose the possible energies for a certain particle are E_1, E_2, E_3, E_4, …, with the values of energy such that $E_1 < E_2 < E_3 < E_4 < \ldots$. Consider a particle in a state given by the following linear superposition of energy basis states:

$$|\phi\rangle = \frac{1}{\sqrt{6}}|E_1\rangle + \frac{2i}{\sqrt{6}}|E_3\rangle + \frac{1}{\sqrt{6}}|E_4\rangle. \tag{5.3}$$

(a) Calculate the probability of obtaining a value E_3 when the energy of the particle is measured. Calculate similar probabilities for obtaining E_1 and E_4. (b) What is the probability of obtaining a value E_2 in a measurement of the energy of this particle? (c) What is the probability of obtaining a value of either E_1, E_3, or E_4 when measuring the energy of this particle?

Solution: (a) To calculate the probability, we determine the coefficient of the basis state vector $|E_3\rangle$, which is the probability amplitude $c_3 = 2i/\sqrt{6}$, and calculate its magnitude squared,

$$P(E_3) = |c_3|^2 = \left|\frac{2i}{\sqrt{6}}\right|^2 = \left(\frac{2i}{\sqrt{6}}\right)\left(\frac{-2i}{\sqrt{6}}\right) = \frac{2}{3}.$$

In a similar manner, $P(E_1) = 1/6 = P(E_4)$.

(b) Since the basis state vector $|E_2\rangle$ does not appear in the linear superposition, this means that the probability amplitude $c_2 = 0$ and therefore $P(E_2) = |c_2|^2 = 0$. Therefore we would never obtain the value E_2 when measuring the energy of the particle in this state.

(c) Since we want to know the probability of measuring any of these values, we simply add the probabilities for measuring each of these values:

$$P(E_1, E_3, E_4) = P(E_1) + P(E_3) + P(E_4) = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1.$$

This makes sense because these are the only three values of energy that could be measured so that the probability of measuring any one of them is 100%.

In part (c) of the previous example, we see that if the squares of the probability amplitudes in a linear superposition state give us information about probabilities, then the probability amplitudes must satisfy the property

$$|c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + \ldots = 1. \tag{5.4}$$

This condition is referred to as *normalization* and a state vector that satisfies this condition is said to be *normalized*. Therefore we conclude:

State vectors which properly describe the quantum state of a particle must be normalized.

Example 2. Normalization of states.

Consider a particle in a state given by the following linear superposition of energy basis states

$$|\phi\rangle = \frac{1}{\sqrt{6}}|E_1\rangle + \sqrt{\frac{3}{6}}|E_2\rangle + c_3|E_3\rangle. \tag{5.5}$$

Determine some possible values for the probability amplitude c_3 such that this state is properly normalized.

Solution: We use the condition given in equation 5.4

$$\begin{aligned} |c_1|^2 + |c_2|^2 + |c_3|^2 &= 1 \\ \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\sqrt{\frac{3}{6}}\right|^2 + |c_3|^2 &= 1 \end{aligned}$$

which is satisfied for the value $|c_3|^2 = 1/3$. The simplest solution for the value of the probability amplitude would be $c_3 = \sqrt{1/3}$. However, the solution $c_3 = i\sqrt{1/3}$ also works because

$$|c_3|^2 = \left|i\sqrt{\frac{1}{3}}\right|^2 = \left(i\sqrt{\frac{1}{3}}\right)\left(-i\sqrt{\frac{1}{3}}\right) = -i^2\left(\frac{1}{3}\right) = +\frac{1}{3}.$$

5.3 Quantum Measurements

As discussed earlier, a quantum state $|\phi\rangle$ can be written as a linear superposition of a set of states called a basis. As shown before, this basis could be a set of states each representing a specific energy E_1, E_2, E_3, ... The same quantum state $|\phi\rangle$ could also be written in terms of a basis of states having definite values of some other measurable quantity, say momentum (p_1, p_2, p_3, ...). Thus the same state could be written as either

$$|\phi\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle + \cdots \qquad \text{(Energy)} \qquad (5.6)$$

or

$$\phi\rangle = d_1|p_1\rangle + d_2|p_2\rangle + d_3|p_3\rangle + \cdots . \qquad \text{(Momentum)} \qquad (5.7)$$

Which linear superposition we use depends upon which quantity we want to measure. If we are interested in doing an experiment in which we measure the energy of the particles, we would therefore use equation (5.6) to compute the probabilities for obtaining the allowed energy values $E_1, E_2, E_3, \ldots$ from the probability amplitudes c_1, c_2, c_3, ...

In quantum mechanics we consider the measurement of a quantity (say energy E) associated with a particle in the state $|\phi\rangle$ in the following way, as indicated in Fig. 5.1. First, a particle is prepared in some quantum state $|\phi\rangle$ and then is sent into some device (which I have called an *Energy Device* in the figure) which is capable of measuring the particle's energy. As a result of the measurement, the device displays the value obtained in the measurement. The value displayed by the device can be *only* one of the discrete values E_1, E_2, E_3, ..., and nothing else! Before the measurement is made, we cannot predict what the resulting value of measured energy will be for a particle in a state such as given by Eq. (5.6). We can only calculate the probability of getting a particular value. For instance, Fig. 5.1 shows the result of a particular measurement yielding the energy value E_2. The probability of obtaining this result is $P(E_2) = |c_2|^2$.

Quantum mechanics also has something new to say about the state of the particle *after* the measurement is made. In terms of classical physics, the measurement leaves the system untouched and the system is left in the

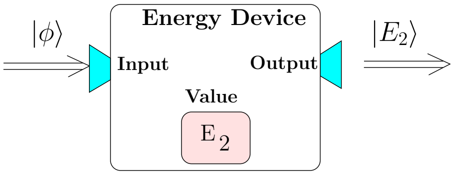


Figure 5.1: Conceptual device for measuring the energy of a particle in a state $|\phi\rangle$. In this example, the result of the measurement is the energy value E_2 and the state of the particle collapses to $|E_2\rangle$.

same state as it was before the measurement. On the other hand, quantum mechanics says something completely different:

> **Collapse of the State:** An ideal measurement of the state of a system forces the system, at the instant of measurement, into a particular basis state vector corresponding to the measured value.

As indicated in Fig. 5.1, if a particle is in the state $|\phi\rangle$ and a measurement of the energy results in obtaining a value of E_2, then the state of the system "collapses" into the state $|E_2\rangle$. In general, **measurements made on quantum systems affect — and often change — the state of the system.** This is a profound statement for it says that the particle being measured and the measurement device can not be treated independently. The measurement device must be considered as part of the system being measured!

5.4 Spin

We would now like to introduce a new property of particles that lends itself nicely to being described using the new mathematical formalism that we have just introduced. In addition to the angular momentum associated with a particle's motion about some origin, many types of elementary particles have an *intrinsic* angular momentum, as though the particle were a tiny sphere rotating about an internal axis. This internal angular momentum is given the name *spin* and is represented by the vector symbol $\vec{S}$. Every particle is characterized by a spin quantum number, s, and the magnitude and z-component of spin angular momentum are given by

$$|\vec{S}| = \sqrt{s(s+1)}\hbar, \tag{5.8}$$

$$S_z = m_s\hbar. \tag{5.9}$$

An important issue for spin angular momentum is the allowed values associated with the quantum numbers s. These values can be determined from a complicated mathematical analysis whose details are beyond the level of this class. However, the interesting result is that the possible values for spin quantum numbers are integer multiples of 1/2, i.e., $s = 0$, 1/2, 1, 3/2, 2, 5/2, …, which differ from the allowed values of l because the list now includes half-integer values. Electrons, protons, neutrons, and ^{3}He nuclei all have the same spin quantum number $s = 1/2$ (sometimes referred to as "spin-$\frac{1}{2}$"); photons have $s = 1$ (or "spin-1"); He^4 nuclei have $s = 0$ (or "spin-0").

Every kind of particle has its particular value of s — for example electrons have $s = 1/2$. This means that every electron in the universe has *exactly* the same intrinsic angular momentum magnitude: $|\vec{S}| = (\sqrt{3}/2)\hbar$. For classical, macroscopic-sized objects, we can change the spin angular momentum by changing the rotation rate of the object. However, in quantum mechanics we are stuck with a constant magnitude of spin angular momentum for elementary particles.

All elementary particles are classified according to their spin quantum numbers s. All particles with integer spin quantum numbers ($s = 0$, 1, 2, …) are called *bosons*, and all particles with half-odd integer spin quantum numbers ($s = 1/2$, 3/2, 5/2, …) are called *fermions*. This distinction is crucial in determining the behavior of systems consisting of many of these particles, as will be discussed in later chapters.

Equation (5.9) gives the possible values one can measure for the component of spin in the z-direction, S_z. For spin quantum number s, the possible values of S_z are given by $m_s\hbar$, where the quantum number m_s can assume values

$$m_s = -s,\ -s+1,\ -s+2,\ \ldots,\ s-2,\ s-1,\ s. \tag{5.10}$$

For a spin–1/2 particle, for instance, S_z can be $-\hbar/2$ or $+\hbar/2$. For a spin-1 particle, S_z can be $-\hbar$, 0 or $+\hbar$. For a spin-3/2 particle, S_z can be $-3\hbar/2$, $-\hbar/2$, $+\hbar/2$, or $+3\hbar/2$.

The points raised in the previous paragraphs should make your head spin (no pun intended)[1]. There are a few issues here that are inherently quantum in nature:

(1) The word *spin* glosses over just how strange this phenomena actually is. It is easy to envision, say, a basketball that is spinning on its axis, resulting in the ball having angular momentum. However, elementary particles can have spin angular momentum *even if the particle has no spatial extent!* The electron, for instance, is a point-particle (as far as our best measurements can determine), as are quarks, and yet these particles carry angular momentum. A photon doesn't even have any mass and yet it, too, carries angular momentum.

---

[1] Okay, so maybe the pun isn't *entirely* unintentional.

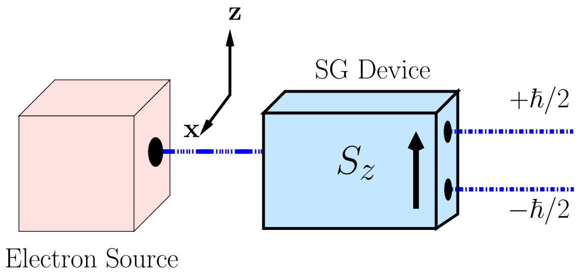


Figure 5.2: Stern-Gerlach apparatus for measuring the z-component of spin.

(2) A spinning basketball could always be stopped, such that its angular momentum becomes zero. This is not the case for an elementary particle. An electron can't be stopped from "spinning" — the spin angular momentum is always there.

(3) Although spin angular momentum is a vector with both a magnitude and direction, measuring *components* of that vector along different directions produces results that simply can't be explained classically. For instance, if a basketball were spinning, you could define a direction for the angular momentum $\vec{S}$. Measurement of the component of the angular momentum perpendicular to $\vec{S}$ would produce a value of zero. This is often not the case for spin angular momentum for elementary particles. For instance, for an electron it is *impossible* to obtain a value of zero from *any* measurement of *any* component of spin angular momentum.

To illustrate some of these issues, we will introduce a device capable of measuring the z-component of spin, S_z, for a particle. This device is called a "Stern-Gerlach" (SG) device, named after the two physicists who first performed experiments like those we describe. Although the technical details of the device are not important to our discussion, the significant thing is that this device is capable of separating a beam of particles according to the z-component of their spins, where the z-axis is determined by the orientation of the SG device.

If we take a source of electrons with random spin component (say from a heated filament) and direct a beam of these electrons through a SG device to measure S_z, as shown in Fig. 5.2, we find that the SG device splits the beam into two beams: one composed of electrons with $S_z = +\hbar/2$ (called "spin-up") and the other with $S_z = -\hbar/2$ (called "spin-down"). Approximately equal numbers of electrons appear in the two beams.

If we now direct the spin-up beam from the SG device to a second SG device, as shown in Fig. 5.3, the second SG device outputs only a spin-up beam ($S_z = +\hbar/2$) and *no* spin-down beam. What does this mean?

Obviously if you measure $S_z = +\hbar/2$ in the first SG device, then if you measure S_z again you will always get the same result. This means that all of the electrons entering the second SG device are spin-up electrons, or are in the spin-up "state."

Now imagine that we take the spin-up beam from the first SG device and direct it to a second SG device which measures the x-component of spin S_x, as in Fig. 5.4. This can be accomplished by orienting a SG device along the x-axis as shown. What will be the result of the measurement made by this second SG device?

The measurement of S_x results in two beams with approximately equal count rates, one of $S_x = +\hbar/2$ (spin-up along x) and the other of $S_x = -\hbar/2$ (spin-down along x). This result is true regardless of whether we had used the spin-up or spin-down beam from the first SG device. Also if we send the electrons into the second SG device one at a time, what we observe from the output is an electron randomly coming out of the second SG device as either $S_x = +\hbar/2$ or $S_x = -\hbar/2$. So the result of this experiment is that we have measured S_z to be $+\hbar/2$ for an electron in the first SG device and S_x to be either $+\hbar/2$ or $-\hbar/2$ for the same electron in the second SG device.

Now we perform a very interesting experiment. As shown in Fig. 5.5, we select electrons in the state spin-up along z in the first SG device and then select electrons in the state spin-down along x in the second SG device. If we now pass these electrons through a third SG device to measure S_z once again, what will be the result? Shouldn't we get just all electrons coming out in the spin-up ($S_z = +\hbar/2$) beam, since that is what we measured in the first SG device?

The answer is — *"NO!"* The measurement of S_z in the third SG device shows electrons coming out as either spin-up ($S_z = +\hbar/2$) or spin-down ($S_z = -\hbar/2$) with equal count rate. This certainly is not the classically expected result! It is as if the electrons lost all memory of being in the spin-up state from the first SG device.

This stuff should really bother you! There is simply no classical way to explain the observations (and these results are from *experimental observations*) about spin and its components. But that's the way in which

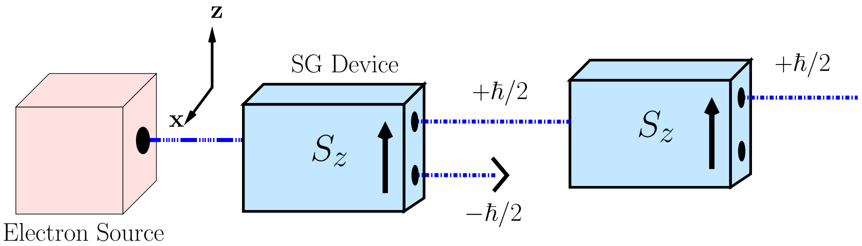


Figure 5.3: Consecutive measurements of z-component for spin-up electrons.

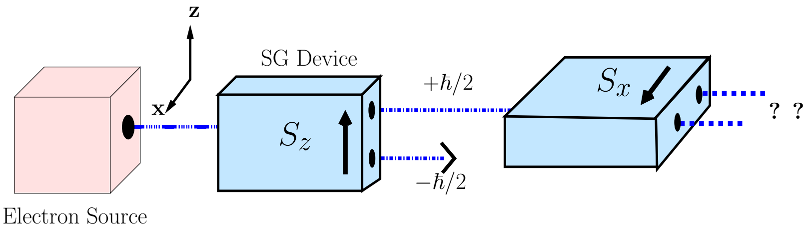


Figure 5.4: Measurement of S_x after measuring S_z.

sub-atomic particles work.

5.5 State Representation for Spin

We can now apply the mathematical framework of state vectors from the previous sections to the case of spin states for particles. When considering the spin state of an electron, the component of spin measured along a certain direction has only two possible states, either "spin-up" or "spin-down"; i.e., with a component of either $+\hbar/2$ or $-\hbar/2$. For the component of spin in the z-direction, we can denote the "spin-up" and "spin-down" states by the kets $|{+z}\rangle$ and $|{-z}\rangle$, respectively. For example, in Fig. 5.2, the electrons exiting the top and bottom of the SG device are in states $|{+z}\rangle$ and $|{-z}\rangle$, respectively. Measurement of the z-component of spin S_z for the state $|{+z}\rangle$ will always produce the value $+\hbar/2$ (as indicated by the experiment in Fig. 5.3), and the same measurement for the $|{-z}\rangle$ state will always produce a value $S_z = -\hbar/2$. Similarly, we can define the states $|{+x}\rangle$ and $|{-x}\rangle$ as the states with x-component of spin $S_x = +\hbar/2$ and $S_x = -\hbar/2$, respectively. Similar definitions apply for the states $|{+y}\rangle$ and $|{-y}\rangle$.

In the experiment depicted in Fig. 5.5 we found that a measurement of the z-component of spin for an electron in the state $|{+x}\rangle$ will produce either $+\hbar/2$ or $-\hbar/2$. This implies that it should be possible to write $|{+x}\rangle$ as a superposition of the states $|{+z}\rangle$ and $|{-z}\rangle$:

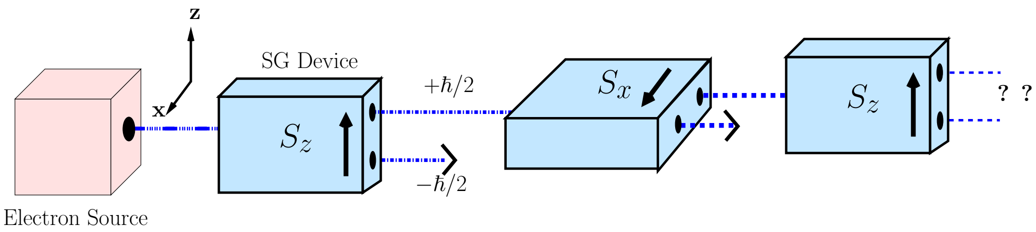


Figure 5.5: Measurement of S_z after measuring S_x.

$$|{+x}\rangle = c_+|{+z}\rangle + c_-|{-z}\rangle \tag{5.11}$$

where c_+ and c_- are constants. We interpret this equation in the following way: if an electron is in a state $|{+x}\rangle$ and we measure the z-component of spin S_z, then the probabilities that the measurement will yield the result $+\hbar/2$ or $-\hbar/2$ are $|c_+|^2$ and $|c_-|^2$, respectively.

Since the results of the experiment in Fig. 5.5 give beams of equal intensity from the third SG device measuring S_z, this means that $|c_+|^2 = |c_-|^2 = 1/2$. Therefore the simplest linear superposition of states representing $|{+x}\rangle$ consistent with this result is

$$|{+x}\rangle = \sqrt{\frac{1}{2}}|{+z}\rangle + \sqrt{\frac{1}{2}}|{-z}\rangle. \tag{5.12}$$

The same can be done for the states $|{-x}\rangle$, $|{+y}\rangle$, and $|{-y}\rangle$. The linear superpositions for these states in terms of the states $|{+z}\rangle$ and $|{-z}\rangle$ are presented below without proof:

$$\begin{aligned}
|{+x}\rangle &= \sqrt{\frac{1}{2}}\,|{+z}\rangle + \sqrt{\frac{1}{2}}\,|{-z}\rangle && \text{(5.13a)}\\
|{-x}\rangle &= \sqrt{\frac{1}{2}}\,|{+z}\rangle - \sqrt{\frac{1}{2}}\,|{-z}\rangle && \text{(5.13b)}\\
|{+y}\rangle &= \sqrt{\frac{1}{2}}\,|{+z}\rangle + i\sqrt{\frac{1}{2}}\,|{-z}\rangle && \text{(5.13c)}\\
|{-y}\rangle &= \sqrt{\frac{1}{2}}\,|{+z}\rangle - i\sqrt{\frac{1}{2}}\,|{-z}\rangle. && \text{(5.13d)}
\end{aligned}$$

Don't forget the factors of "i" in the equations for $|{+y}\rangle$ and $|{-y}\rangle$; they are important.

Example 3. Spin-up and spin-down states of S_z in terms of spin-up and spin-down states of S_x.

Using Equations 5.13a and 5.13b, show that we can write the spin-up S_z state $|{+z}\rangle$ as a linear superposition of the states $|{+x}\rangle$ and $|{-x}\rangle$.

Solution: If we add equations 5.13a and 5.13b we obtain

$$\begin{aligned}
|{+x}\rangle + |{-x}\rangle &= \left[\sqrt{\frac{1}{2}}|{+z}\rangle + \sqrt{\frac{1}{2}}|{-z}\rangle\right] + \left[\sqrt{\frac{1}{2}}|{+z}\rangle - \sqrt{\frac{1}{2}}|{-z}\rangle\right]\\
&= 2\cdot\sqrt{\frac{1}{2}}|{+z}\rangle.
\end{aligned}$$

Solving for $|{+z}\rangle$ we get

$$|{+z}\rangle = \sqrt{\frac{1}{2}}\,|{+x}\rangle + \sqrt{\frac{1}{2}}\,|{-x}\rangle.$$

Similarly, if we subtract equations 5.13a and 5.13b and solve for $|{-z}\rangle$ we obtain

$$|{-z}\rangle = \sqrt{\frac{1}{2}}|{+x}\rangle - \sqrt{\frac{1}{2}}|{-x}\rangle.$$

In a similar manner we can use equations 5.13a $-$ 5.13d to determine any of the basis spin vectors in terms of the others. The complete set of transformations among the basis spin states appears in Table 5.1.

The previous example demonstrates exactly what we discovered in the experiment of Fig. 5.4. If we have a beam of electrons in the state $|{+z}\rangle$ and measure the x-component of spin S_x, we find that we obtain $S_x = +\hbar/2$ (spin-up along x) and $S_x = -\hbar/2$ (spin-down along x) with equal probabilities $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

Example 4. Spins and probabilities.

Consider an electron in the state given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|{+z}\rangle + \sqrt{\frac{1}{3}}|{-z}\rangle. \tag{5.14}$$

(a) Calculate the probability of obtaining the value $+\hbar/2$ and $-\hbar/2$ when the z-component of the spin angular momentum is measured.

Table 5.1: Transformation of the basis spin vectors.

$$\begin{aligned}
|{+z}\rangle &= \sqrt{\tfrac{1}{2}}|{+x}\rangle + \sqrt{\tfrac{1}{2}}|{-x}\rangle = \sqrt{\tfrac{1}{2}}|{+y}\rangle + \sqrt{\tfrac{1}{2}}|{-y}\rangle \\
|{-z}\rangle &= \sqrt{\tfrac{1}{2}}|{+x}\rangle - \sqrt{\tfrac{1}{2}}|{-x}\rangle = -i\sqrt{\tfrac{1}{2}}|{+y}\rangle + i\sqrt{\tfrac{1}{2}}|{-y}\rangle \\
|{+x}\rangle &= \sqrt{\tfrac{1}{2}}|{+z}\rangle + \sqrt{\tfrac{1}{2}}|{-z}\rangle \\
|{-x}\rangle &= \sqrt{\tfrac{1}{2}}|{+z}\rangle - \sqrt{\tfrac{1}{2}}|{-z}\rangle \\
|{+y}\rangle &= \sqrt{\tfrac{1}{2}}|{+z}\rangle + i\sqrt{\tfrac{1}{2}}|{-z}\rangle \\
|{-y}\rangle &= \sqrt{\tfrac{1}{2}}|{+z}\rangle - i\sqrt{\tfrac{1}{2}}|{-z}\rangle
\end{aligned}$$

(b) Calculate the probability that an electron in the state $|\psi\rangle$ will be measured to have an x-component of spin of $-\hbar/2$. (c) Calculate the probability that an electron in the state $|\psi\rangle$ will be measured to have a y-component of spin of $-\hbar/2$.

Solution: (a) This is straightforward because $|\psi\rangle$ is written as a superposition of the $|\pm z\rangle$-states. The probability of the spin being found as "spin up" is the square of the coefficient of the base state $|+z\rangle$. That is,

$$\text{Prob}\big(\text{spin measured to be spin-up}\big) = P(+z) = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}.$$

Likewise, the probability of the spin being found as "spin down" is

$$\text{Prob}\big(\text{spin measured to be spin-down}\big) = P(-z) = \left|\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{3}.$$

(b) For the x-component of spin, we need to rewrite $|\psi\rangle$ as a linear superposition of the S_x basis states $|+x\rangle$ and $|-x\rangle$. This can be accomplished by using the results of Example 3 and Table 5.1 to write the states $|+z\rangle$ and $|-z\rangle$ in terms of the basis states $|+x\rangle$ and $|-x\rangle$.

$$\begin{aligned}
|\psi\rangle &= \sqrt{\frac{2}{3}}\left[\sqrt{\frac{1}{2}}\,[|+x\rangle + |-x\rangle]\right] + \sqrt{\frac{1}{3}}\left[\sqrt{\frac{1}{2}}\,[|+x\rangle - |-x\rangle]\right] \\
&= \sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}}\right]|+x\rangle + \sqrt{\frac{1}{2}}\left[\sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}}\right]|-x\rangle \\
&= \left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}}\right)|+x\rangle + \left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right)|-x\rangle. \qquad (5.15)
\end{aligned}$$

Therefore, the probability that the electron is in the spin-down state of S_x is equal to the square of the coefficient of the $|-x\rangle$ basis state

$$\text{Prob}\big(\text{spin-down in } S_x\big) = \left|\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right|^2 \approx 0.029. \qquad (5.16)$$

(c) The calculation here is very similar to that for part (b) *except* that we should write the states $|\pm z\rangle$ in terms of the basis states $|\pm y\rangle$. Since we are interested in measuring a value of the y-component of spin, S_y, we need to express our quantum state as a superposition of the S_y basis states. We will leave this to you to finish, but you should obtain an answer of 0.5 for the probability.

5.6 Spin Magnetic Moment in a Magnetic Field

We are now ready to apply the major ideas of this chapter to some interesting examples which have important applications to medical physics, chemistry, and astronomy.

Both protons and electrons have intrinsic spin, with spin quantum number $s = 1/2$ (both are fermions). The proton also has a charge $q = +e$. Classically speaking, the proton is viewed as a tiny spinning charged sphere that has current loops associated with the circulating charge. In the previous unit on electricity and magnetism, we learned that a magnetic moment $\vec{\mu}$ can be associated with these circulating currents. When these concepts are applied to the proton as a classical spinning charged sphere, we find that the magnetic moment of the proton is proportional to the proton's intrinsic angular momentum $\vec{S}$. In fact, the magnetic moment for the proton is found to be

$$\vec{\mu} = \frac{2\mu_{\mathrm{p}}}{\hbar}\vec{S} \tag{5.17}$$

where $\mu_{\mathrm{p}} = 1.41\times10^{-26}$ J/T. Since measuring any component of $\vec{S}$ results in a value of $\pm\frac{\hbar}{2}$, equation 5.17 tells us that, **for a proton**, $\vec{\mu}$ has a magnitude of μ_{p} and points in the direction of $\vec{S}$.

We learned in the unit on electricity and magnetism that if a magnetic moment $\vec{\mu}$ is placed in a constant magnetic field $\vec{B}$, there is a magnetic potential energy associated with the orientation of the magnetic moment relative to the magnetic field given as

$$U = -\vec{\mu}\cdot\vec{B}. \tag{5.18}$$

This energy is greatest when $\vec{\mu}$ and $\vec{B}$ are pointing in opposite directions and least when pointing in the same direction. If we take $\vec{B}$ to be pointing in the z-direction such that $\vec{B} = B_0\hat{z}$, then Eqs. 5.17 and 5.18 can be written as

$$U = -\vec{\mu}\cdot\vec{B} = -\frac{2\mu_{\mathrm{p}}}{\hbar}B_0 S_z = -2\mu_{\mathrm{p}}B_0 m_s \tag{5.19}$$

where we have used equation (5.9) for the z-component of the proton's spin. Since $m_s = \pm1/2$ for a spin-1/2 particle, each of the orientations of the proton (spin-up or spin-down) corresponds to a different energy of the proton in the magnetic field

$$U_+ = -\mu_{\mathrm{p}}B_0 \quad (m_s = +\tfrac{1}{2}, \text{ spin-up}) \tag{5.20}$$

$$U_- = +\mu_{\mathrm{p}}B_0 \quad (m_s = -\tfrac{1}{2}, \text{ spin-down}). \tag{5.21}$$

Consequently, the proton has a different energy depending on whether it is in the spin-up ($|{+z}\rangle$) state or the spin-down ($|{-z}\rangle$) state. Therefore the states $|{+z}\rangle$ and $|{-z}\rangle$, in addition to being states of definite z-component of spin

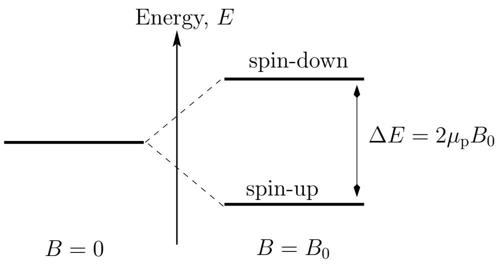


Figure 5.6: Splitting of energy levels of a **proton** in a magnetic field

S_z, are also states of definite energy U_+ and U_-, respectively. Thus when placed in a magnetic field, the state of the proton splits into two possible energy states, as shown in Fig. 5.6 separated in energy by

$$\Delta E_\mathrm{proton} \equiv |U_+ - U_-| = 2\mu_\mathrm{p} B_0. \tag{5.22}$$

Figure 5.6 suggests that for a spin-up proton $|+z\rangle$ placed in a magnetic field $\vec{B} = B_0\hat{z}$, if the proton then absorbs a photon of energy $E_\mathrm{photon} = \Delta E_\mathrm{proton} = 2\mu_\mathrm{p} B_0$, then the proton will make a transition to the spin-down state $|-z\rangle$, which has higher energy. The reversal of spin component S_z is called *spin-flipping.*

Example 5. Spin-flip transitions of protons in a magnetic field.

Consider a proton that is placed in a magnetic field such that it is in the spin-up state $|+z\rangle$. If the magnitude of the magnetic field is $B_0 = 24.8\,\mathrm{mT}$, determine the frequency of the photon that would be absorbed by the proton causing it to make a transition to the spin-down state?

Solution: For the magnetic field strength given, the energy difference between the spin-up and spin-down states is

$$\begin{aligned}\Delta E_\mathrm{proton} &= 2\mu_\mathrm{p} B_0 \\ &= 2(1.41\times 10^{-26}\,\mathrm{J/T})(24.8\times 10^{-3}\,\mathrm{T}) \\ &\approx 6.99\times 10^{-28}\,\mathrm{J}.\end{aligned} \tag{5.23}$$

Therefore the frequency associated with a photon of this energy is

$$f_\mathrm{photon} = \frac{E_\mathrm{photon}}{h} = \frac{\Delta E_\mathrm{proton}}{h} = \frac{6.99\times 10^{-28}\,\mathrm{J}}{6.63\times 10^{-34}\,\mathrm{J{\cdot}s}} = 1.05{\times}10^{6}\,\mathrm{Hz} = 1.05\,\mathrm{MHz}.$$

This frequency lies in the region of the electromagnetic spectrum associated with radio waves.

Like the proton, an electron also has intrinsic spin with spin quantum number $s = 1/2$. In contrast to the proton, the negative charge of an electron means its magnetic moment $\vec{\mu}$ points opposite the direction of $\vec{S}$. The electron has $\mu_\mathrm{e} = -9.28 \times 10^{-24}\,\mathrm{J/T}$, where the negative sign denotes the opposing directions of $\vec{\mu}$ and $\vec{S}$. Replacing μ_p with μ_e in equation 5.19, we find that the higher energy level for an electron occurs when the electron is spin-up and the lower energy level occurs when the electron is spin-down (for the case where $\vec{B}$ is pointing in the positive z-direction).

In a hydrogen atom, the nucleus (a proton) and the electron both have spin magnetic moments. Thus, the energy levels of the hydrogen atom are slightly split by the interaction between the electron spin and the nuclear spin, a phenomenon known as *hyperfine splitting*. When the electron's spin flips, the atom transitions between two hyperfine energy levels and a photon can be emitted or absorbed in the process. Because so much of the gas composing galaxies is neutral hydrogen, this spin-flip transition allowed astronomers to map the spiral structure of the Milky Way for the first time. In your homework, you'll take a closer look at hyperfine splitting.

Problems

1. Write either a poem, a song, or a few sentences explaining what it is about spin — on a quantum level — that can't be explained by classical laws of physics.

2. Suppose an electron is known to have a z-component of spin S_z of $+\hbar/2$; that is, we know it is in the $|+z\rangle$ state.

(a) Find the probability that a measurement of its spin along the x-axis gives the value $+\hbar/2$.

(b) Find the probability that a measurement of its spin along the y-axis gives the value $-\hbar/2$.

3. Assume that a particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + i\frac{1}{\sqrt{2}}|\phi_2\rangle$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are a normalized set of basis states.

(a) A measurement is made on the particle. Calculate the probability that this measurement will result in a value corresponding to the state $|\phi_1\rangle$.

(b) Let's say that the measurement made in part (a) *does*, in fact, produce a result corresponding to the state $|\phi_1\rangle$. The measurement is then immediately repeated. Calculate the probability that this *second* measurement will result in a value corresponding to the state $|\phi_2\rangle$.

(c) Another particle is prepared in the original state $|\psi\rangle$. A measurement is made on this new particle. Calculate the probability that *this* measurement will result in a value corresponding to the state $|\phi_2\rangle$.

4. An electron is known to be in the spin state $|\psi\rangle = \frac{3}{5}|+z\rangle + \frac{4}{5}|-z\rangle$.

(a) The electron is sent through a device that measures its spin angular momentum along the x-direction. Compute the probability of obtaining the result $+\hbar/2$ for the x-component of spin. Compute the probability of obtaining $-\hbar/2$ for this measurement. (Check to make sure that your probabilities add up to 1.)

(b) The electron initially in the state $|\psi\rangle$ specified above is sent through a device that measures its spin angular momentum along the y-direction. Compute the probability of obtaining the result $+\hbar/2$ for the y-component of spin. Compute the probability of obtaining $-\hbar/2$ for this measurement.

(c) Note that you get different answers for parts (a) and (b). Mathematically, this comes from the factor of "i" in the equations for $|+y\rangle$ and $|-y\rangle$. Conceptually, this is another example of some quantum weirdness — you would think that a state that is a superposition of stuff solely in the z-direction would give the same results for $x-$ and y-components, but that isn't the case. Think about this for a moment, and convince yourself that this is weird.

5. Assume that you have an electron that is in a superposition of "spin-up" and "spin-down" states:

$$|\psi_1\rangle = i\sqrt{\frac{1}{5}}|+z\rangle + \sqrt{\frac{4}{5}}|-z\rangle.$$

(a) You measure the vertical component of spin. What is the probability that you will find $S_z = +\hbar/2$?

(b) Another electron is in the same state $|\psi_1\rangle$. You now measure the y-component of spin. What is the probability that you will find $S_y = +\hbar/2$?

(c) Another electron is prepared in a superposition state

$$|\psi_2\rangle = \sqrt{\frac{1}{5}}|+z\rangle + \sqrt{\frac{4}{5}}|-z\rangle.$$

What is the probability that a measurement of *this* electron's y-component of spin will find $S_y = +\hbar/2$?

(d) Now, assume that you do find $S_y = +\hbar/2$. You measure S_y again for the same particle. What is the probability that you'll find $S_y = -\hbar/2$?

6. A particle is in the state given by

$$|\phi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle - \frac{i}{\sqrt{2}}|\psi_2\rangle$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are a normalized set of basis states.

(a) A measurement is made on the particle. Calculate the probability that the measurement will result in a value corresponding to the state $|\psi_1\rangle$.

(b) Another particle is prepared in the state $|\phi\rangle$, and a measurement is made on *this* particle. Calculate the probability that *this* measurement will result in a value corresponding to the state $|\psi_2\rangle$.

7. The quantum state of a photon propagating along the z-direction can be written in terms of the states $|X\rangle$ and $|Y\rangle$, where $|X\rangle$ represents a photon polarized along the x-direction and $|Y\rangle$ represents a photon polarized along the y-direction. A photon in an arbitrary polarization state can be written as a linear superposition of the states $|X\rangle$ and $|Y\rangle$ as

$$|\theta\rangle = \cos\theta|X\rangle + \sin\theta|Y\rangle$$

where θ is the angle of polarization of the photon measured with respect to the x-axis.

(a) Assuming that the states $|X\rangle$ and $|Y\rangle$ form a basis, show that the state $|\theta\rangle$ is normalized.

(b) A polarizer is a device that measures the polarization of a photon along a certain direction. Let's say that a polarizer is oriented along the x-direction, that is, it is measuring for the state $|X\rangle$. For an incident beam of photons in the state $|\theta\rangle$ as given above, what is the probability that the polarizer measures the state $|X\rangle$ (i.e., the photons go through the polarizer)? What is the state of the photons after making this measurement?

(c) Let's say we have a beam of photons in polarization state $|X\rangle$ incident on a polarizer oriented in a direction making an angle θ with respect to the x-axis. What is the probability that the photons pass through the polarizer oriented at angle θ? (Hint: think about the previous question in the reverse order of events). What is the state of the photons after making this measurement?

(d) Following up on part (c), the beam in state $|\theta\rangle$ is incident on a second polarizer oriented along the y-axis. What is the probability that the photons pass through the second polarizer?

(e) Now, putting this all together, photons pass through polarizer #1 oriented along the x-axis, then pass through polarizer #2 oriented at an angle θ, then on to polarizer #3 oriented along the y-axis. What is the total probability that a photon passing through polarizer #1 will also pass through polarizer #3? [Note: Compare your result with what you did in Lab #18 "Polarization of Light."]

8. Using a technique similar to that used in Example 3, show that you can write the spin-down S_z state $|-z\rangle$ as a linear superposition of the states $|+x\rangle$ and $|-x\rangle$ Compare your results with those given in Table 5.1.

9. Using equations (5.13c) and (5.13d), write expressions for the states $|+z\rangle$ and $|-z\rangle$ in terms of linear combinations of the states $|+y\rangle$ and $|-y\rangle$. Compare your results with those given in Table 5.1.

10. An electron is placed in a spin state given by

$$|\psi\rangle = \frac{1}{2}|{+z}\rangle - \frac{\sqrt{3}}{2}|{-z}\rangle.$$

(a) Calculate the probability of obtaining a value of $-\hbar/2$ when the z-component of spin S_z is measured.

(b) Calculate the probability that an electron in state $|\psi\rangle$ will be measured to have an x-component of spin S_x of $+\hbar/2$.

11. An electron is placed in the spin state

$$|\psi\rangle = \sqrt{\frac{2}{3}}|{+z}\rangle + \sqrt{\frac{1}{3}}|{-z}\rangle$$

as is the case in Example 4. An experiment is performed to measure the y-component S_y of spin. Calculate the probability that this measurement results in a value of $-\hbar/2$.

12. An electron in a particular system can have any of the discrete energies $E_n = (-8.00\,\mathrm{eV})/n^2$, where $n = 1, 2, 3, \ldots$. Assume that the electron is in a state

$$|\psi\rangle = 0.7\,|1\rangle + 0.5\,|2\rangle + 0.4\,|3\rangle + 0.3\,|4\rangle + 0.1\,|5\rangle.$$

(a) Show that this state is normalized.

(b) If the energy of the electron is measured, what is the probability that the result of the measurement will be $E_2 = -2.0\,\mathrm{eV}$? What is the probability of the measured energy being $E_4 = -0.5\,\mathrm{eV}$?

(c) What is the probability that the result of measuring the energy would be either E_1, E_3, or E_5?

(d) Assume that 10,000 electrons are prepared to be in the same state $|\psi\rangle$ and that the energy is measured for each of these electrons. Approximately how many of these electrons will be measured to have energy E_1? E_2? E_3? E_4? E_5?

13. Protons placed in a magnetic field can be either in the spin-up $|{+z}\rangle$ or spin-down $|{-z}\rangle$ state with an energy difference between these two states $\Delta E_{\mathrm{proton}}$. Incident photons of frequency 2.20 MHz are absorbed causing transitions of the protons from the spin-up $|{+z}\rangle$ state to the spin-down $|{-z}\rangle$ state. Determine the magnitude of the total magnetic field B in which these protons are placed.

14. The bulk of gas in our galaxy is atomic hydrogen (composed of a proton and an electron) in the ground electronic state. Astronomers map the location of this neutral hydrogen by detecting photons that are emitted when the spin of the **electron** flips from being parallel with the spin of the proton to being anti-parallel. The effective magnetic field experienced by the electron in a hydrogen atom is 0.0507 T. Determine the wavelength of the photon emitted when the electron undergoes a spin flip.

15. In the presence of a strong magnetic field, the energy levels of an atom change due to the interaction of the electronic spins with the magnetic field. This is known as the *"anomalous" Zeeman effect*, and can be used to determine the strength of magnetic fields on the Sun.

(a) Magnetically active regions of the Sun have typical magnetic fields strengths of 0.4 T. Determine the energy difference (in eV) between electrons that are aligned with the magnetic field and anti-aligned with the magnetic field.

(b) This difference in energy causes a split in each electronic energy level of the atom. Imagine that we have a 2-level atom where the 1^{st} excited state is 0.1 eV above the ground state. Draw energy level diagrams for this atom: i) when there is no magnetic field and ii) in the presence of the magnetic field from part (a).

(c) Consider the 2-level atom of part (b) in the absence of a magnetic field. Which photon frequencies could this atom absorb? (This is called the absorption spectrum.)

(d) Now consider the 2-level atom of part (b) in the presence of the 0.4 T magnetic field from part (a). Using your energy level diagram from part (c), determine which photon frequencies this atom could absorb.

(e) How would this spectrum change as the strength of the magnetic field increases?

16. Let's use a simulation of the Stern-Gerlach experiment to explore quantum states and spin. Go to
`http://phet.colorado.edu/sims/stern-gerlach/stern-gerlach_en.html`
By default, the atoms start in the state $|+x\rangle$, and the device is setup to make a measurement of the z-component of spin.

(a) To get a feel for how this works, try firing a few atoms. You'll notice a counter at the bottom of the page keeping track of how many atoms have been measured to be spin up or spin down as measured along the z axis. Can you predict whether an individual atom will have a measured z-component of spin up or down?

(b) For atoms starting in the state $|+x\rangle$, what is the probability that a measurement of the z-component of spin results in $S_z = +\hbar/2$? To determine this, turn on "Auto Fire" and keep it running until the percentages are not longer changing.

(c) For atoms starting in the state $|+x\rangle$, what is the probability that a measurement of the x-component of spin results in $S_x = +\hbar/2$? To measure the x-component of spin, change the "angle" in the simulation to $-90°$.

(d) Now, change the number of magnets to 2. Orient the first magnet so that it is measuring the z-component of spin (angle = 0) and orient the second magnet so that it is measuring the x-component of spin (angle = $-90°$). In this setup the simulation starts with atoms in the state $|+x\rangle$ and measures the z-component of spin. For atoms with measured $S_z = +\hbar/2$, it then measures the x-component of spin. For an atom that started in the state $|+x\rangle$, for which we then measure $S_z = +\hbar/2$, what is the probability that a subsequent measurement of the x-component of spin results in $S_x = +\hbar/2$?

(e) Based on your answer to part (c), what is the state of the atom immediately after a measurement of its z-component of spin results in $S_z = +\hbar/2$? Hint: Think about collapse of state!

17. Suppose the possible energies that a particle can have are E_1, E_2, E_3, E_4, ... Consider a particle in a state given by the following linear superposition of energy basis states:

$$|\psi\rangle = \frac{1+2i}{6}|E_1\rangle + \frac{2-3i}{6}|E_3\rangle + c_4|E_4\rangle, \tag{5.24}$$

where c_4 is an undetermined constant.

(a) Calculate the probability of obtaining a value E_3 in a measurement of the energy of this particle.

(b) The particle is prepared again in the state $|\psi\rangle$. Calculate the probability of obtaining a value E_2 in a measurement of the energy of this particle?

(c) How many possible values are there for the constant c_4? Determine **two** possible values for this constant.

18. A quantum particle is in the following superposition of energy states

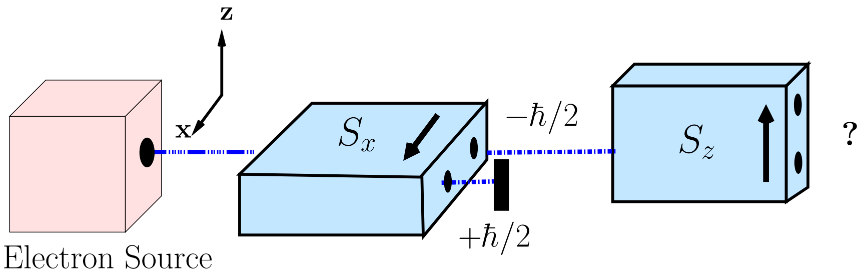
$$|\psi\rangle = \frac{1}{3}\,|E_1\rangle - \frac{4+6i}{9}\,|E_2\rangle - \frac{4i-2}{9}\,|E_3\rangle.$$

(a) Determine the probability that a measurement of the energy will yield the value E_3.

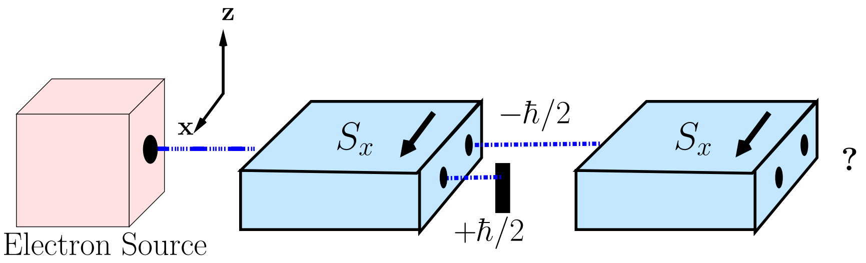
(b) You measure the energy and do, in fact, find the energy to be E_3. Write down a correct expression for the new state of the particle just after this energy measurement.

19. Each of the following diagrams shows a beam of electrons inititally with random spin orientations passing through a series of Stern-Gerlach (SG) apparatuses of different orientations. In each case, determine whether a beam will come out of one, both, or neither of the two exits of the final SG apparatus, and also determine how the intensity of any non-zero beam will compare with the intensity of the initial beam of electrons.

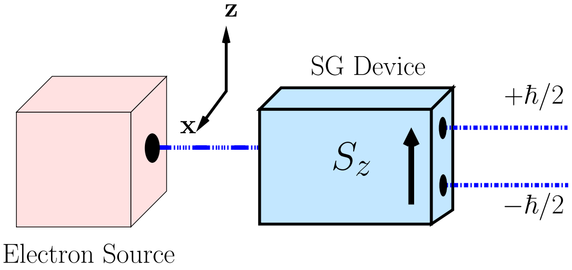
(a)



(b)



20. Consider the following Stern-Gerlach (SG) experiment. A beam of electrons, prepared in a certain state $|\psi\rangle$, is emitted from an electron source. The beam passes into a Stern-Gerlach device which measures the z-component of spin S_z as shown below. A total of 1000 electrons go through this apparatus, of which exactly 400 emerge with $S_z = +\hbar/2$ (spin-up along the z-axis) and exactly 600 emerge with $S_z = -\hbar/2$ (spin-down along the z-axis).



Write a possible normalized state $|\psi\rangle$ for the electrons in the initial beam that corresponds to the observations in this experiment.

Chapter 6

Quantum Statistics and Applications

6.1 Introduction

We have encountered already a number of peculiar features of quantum mechanics such as wave-particle duality, spatially spread out states, and point particles with nonzero spin angular momentum. Now we are going to add another peculiar feature called *quantum indistinguishability* that arises when we make the step up from one particle to two particles.

What we will see is that quantum particles are indistinguishable in a way that classical particles cannot be. And this indistinguishability has very real consequences, ranging from lasers to superconductors to white dwarfs. Most importantly, it is this indistinguishability of quantum particles that leads to the periodic table and the chemical properties of the elements, which, needless to say, play a pretty big role in our everyday existence.

6.2 Quantum Statistics

The whole phenomenon of indistinguishability results from considering pairs of quantum particles and asking the following:

> In a two-particle system where particles 1 and 2 are of the same type, say, both photons or both electrons, is it possible to tell which is particle 1 and which is particle 2?

To be specific, let's consider the ground state of a helium atom. Helium has two electrons orbiting the nucleus and both electrons can be in the ground state energy level. In classical physics, the two electrons *must* be distinguishable: zoom in with a detailed enough probe and you could see which electron is where. You could name them electron-Fred and electron-George, and then follow their motion with a fast camera and keep track of which was Fred and which was George. So the classical physics answer to the question above is yes: with enough technology we can tell which is which.

But the microscopic world isn't classical, and in quantum mechanics things are different. Heisenberg's uncertainty principle has already demanded that the electrons in the ground state have some spread in position and spread in momentum. Since the two electrons are overlapping in this smeared out "Fred-George-blob" state, Heisenberg is prohibiting us from tracking the two particles separately, no matter how fast or high resolution our camera is. Therefore it is at least possible for quantum mechanical particles to be *indistinguishable*. But that just leaves the question open. To demonstrate quantum indistinguishability we will have to work out its implications and then test them against nature.

Let's develop some notation. If particle 1 is in state $|\alpha\rangle$ and particle 2 in state $|\beta\rangle$ (these might be spin states, or energy levels, or whatever), we can write the two-particle state as $|\psi\rangle = |\alpha\,\beta\rangle$. Here we read the particle 1 state as the first symbol and the particle 2 state as the second symbol.

But if these quantum particles are indistinguishable we cannot really have the state $|\alpha\,\beta\rangle$, since that notation says we know it's particle 1 in state $|\alpha\rangle$, and by assumption we can't know this. What we really want to say is "one of the particles is in $|\alpha\rangle$ and the other particle is in $|\beta\rangle$ and that's all we know." We can achieve this by making a superposition:

$$|\psi_S\rangle = \frac{1}{\sqrt{2}}|\alpha\,\beta\rangle + \frac{1}{\sqrt{2}}|\beta\,\alpha\rangle. \tag{6.1}$$

The state $|\psi_S\rangle$ truly makes no distinction between particle 1 and particle 2. You can see this by consider the effect of swapping the particle labels 1 and 2:

$$|\alpha\,\beta\rangle \xrightarrow[\text{swap } 1\leftrightarrow 2]{} |\beta\,\alpha\rangle. \tag{6.2}$$

If we apply this "swap" operation on $|\psi_S\rangle$, we find that we get the same state back:

$$|\psi_S\rangle \xrightarrow[\text{swap } 1\leftrightarrow 2]{} \frac{1}{\sqrt{2}}|\beta\,\alpha\rangle + \frac{1}{\sqrt{2}}|\alpha\,\beta\rangle = |\psi_S\rangle. \tag{6.3}$$

Therefore the state $|\psi_S\rangle$, which is the *symmetric* superposition of $|\alpha\,\beta\rangle$ and $|\beta\,\alpha\rangle$, is a candidate state to describe indistinguishable particles.[1]

There is another possibility, though. Recall that the overall sign of a wavefunction or state has no impact on any measurable quantity. All the measurements we can make amount to probabilities, which come from squaring the coefficients in front of the states. So we could also make indistinguishable particle states with the *antisymmetric* combination

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\,\beta\rangle - \frac{1}{\sqrt{2}}|\beta\,\alpha\rangle. \tag{6.4}$$

---

[1] The $1/\sqrt{2}$ coefficients are required for normalization.

This state picks up exactly an overall minus sign under the “swap” operation:

$$|\psi_A\rangle \xrightarrow[\text{swap } 1\leftrightarrow 2]{} \frac{1}{\sqrt{2}}|\beta\,\alpha\rangle - \frac{1}{\sqrt{2}}|\alpha\,\beta\rangle = -|\psi_A\rangle. \tag{6.5}$$

So we have two possibilities for quantum indistinguishability: the two-particle states can either be the symmetric $|\psi_S\rangle$ or the antisymmetric $|\psi_A\rangle$. And now for a strong statement.

> Every particle that has ever been discovered exhibits quantum indistinguishability!

This includes photons, electrons, protons, neutrons, the quarks that make up protons and neutrons, and more exotic particles like the W and Z bosons. It seems to be a fundamental property of the quantum world obeyed by everything, just like wave-particle duality.

The particles that are symmetric under swap are called *bosons*,[2] and the particles that are antisymmetric under swap are called *fermions*.[3] We will consider both cases in detail below.

6.3 Relation between Statistics and Spin

All fundamental particles are either fermions or bosons. Strangely enough, which category they fall into is directly determined by their value of spin.

- All particles with spin $s = 0, 1, 2, \ldots$ or integer spin, are bosons: their two-particle states must be symmetric. This includes the photon and some more exotic particles that we will encounter in Unit 4.
- All particles with spin $s = 1/2, 3/2$, etc, which we call half-integer spin, are fermions: their two-particle states must be antisymmetric. These include electrons, protons, neutrons, and quarks, and more.

The connection between the spin of a particle and its symmetry under exchange (swap) does not have a simple origin, and we will not try to explain it here. But it is another rule strictly obeyed by every particle.

Note that in some cases a collection of particles can act as a single particle. For example, a proton is really made up of three quarks. Each quark is a fermion, so if we imagine exchanging a pair of protons, we are really exchanging three pairs of quarks. Each quark exchange brings in a minus sign, so the net effect of exchanging the protons in a two-proton state is a factor of $(-1)^3 = -1$. Thus, the proton will act like a fermion, as long

---

[2] In honor of the Indian physicist Satyendra Bose.

[3] In honor of the Italian-American physicist Enrico Fermi.

as it makes sense to think of the proton as a unit, that is, as long as we are not considering some process that rips apart the protons.

A similar game can be played with entire atoms. A helium-4 atom consists of two protons and two neutrons in the nucleus, orbited by two electrons. That is an even number of fermions, so if we make a two-atom state out of two helium atoms, they will be symmetric under exchange of the atoms. So the helium-4 atom, if it stays intact, acts as a boson.

6.4 Fermions and the Pauli Exclusion Principle

A very interesting thing happens to fermions in the antisymmetric two-particle state when the single-particle states $|\alpha\rangle$ and $|\beta\rangle$ are taken to be the same state. Then we have

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}|\alpha\,\alpha\rangle - \frac{1}{\sqrt{2}}|\alpha\,\alpha\rangle = 0. \tag{6.6}$$

What does this mean? It means that there is no such state! This is known as the Pauli exclusion principle, which can be stated as follows:

Pauli exclusion principle: it is not possible to put two identical fermions into the same single-particle state.

Let's see what it does in practice. Electrons are fermions. Consider an electron in an infinite square-well potential (particle-in-a-box) in the lowest energy level E_1. There are two possible states: $|E_1\!\uparrow\rangle$ and $|E_1\!\downarrow\rangle$, meaning "ground state, spin up" and "ground state, spin down."

In the ground state, there are two electrons in this lowest energy level. If we attempt to put these two electrons in both with spin up we get

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}|E_1\!\uparrow\;\; E_1\!\uparrow\rangle - \frac{1}{\sqrt{2}}|E_1\!\uparrow\;\; E_1\!\uparrow\rangle = 0. \tag{6.7}$$

We find that the state doesn't exist! The Pauli exclusion principle tells us we cannot put the two electrons into the ground state with the spins both up.

But we can put the two electrons into the ground state when their spins are opposite. That is, the real ground state is

$$|\text{He g.s.}\rangle = \frac{1}{\sqrt{2}}|E_1\!\uparrow\;\; E_1\!\downarrow\rangle - \frac{1}{\sqrt{2}}|E_1\!\downarrow\;\; E_1\!\uparrow\rangle. \tag{6.8}$$

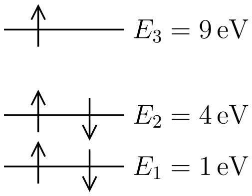
This expression has the proper anti-symmetry and is not zero.

This raises the question of whether we could put three electrons into the ground state. As you might guess, there is no three-electron antisymmetric state $|\psi_A\rangle$, so the upper limit on the number of electrons we can put into the ground state is two.

Example 1. A Five Particle System

Five identical particles with spin $s = 1/2$ are put into a one-dimensional infinite square well with energy levels $E_n = n^2 E_1$ and $E_1 = 1\,\text{eV}$. What is the lowest combined energy that these particles can have?

Solution: Two particles can fit into the lowest energy level, in an antisymmetric combination like Eq. (6.8). Two more particles can fit into the second lowest energy level, and then one particle must be in the third energy level. This is illustrated schematically on the right. Adding up these energies:



$$E_{\text{combined}} = 1 + 1 + 4 + 4 + 9 = \boxed{19\,\text{eV.}}$$

Pauli's Exclusion Principle has **tremendously** important implications for the behavior of atoms and for the entire field of chemistry (and, therefore, biology as well). We'll discuss these implications in the next chapter.

6.5 Bosons

Now let's turn to bosons, the particles that form symmetric combinations $|\psi_S\rangle$ in their two-particle states. These particles do not exhibit a Pauli exclusion principle. In fact, it's quite the opposite: bosons like to be in the same state, relative to distinguishable particles.[4]

Demonstrating this effect is a little bit tricky. Let us consider the following scenario: imagine that there are three different states, which we'll call $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$, and there are two particles which will be put into those states. How many possible two-particle states are there?

For classical distinguishable particles, each particle can be in one of three possible states, so there are $3 \times 3 = 9$ possible two-particle states. These are illustrated in the chart on the left in Fig. 6.1. Since the particles are distinguishable, we use different symbols for them: "•" for particle 1 and "○" for particle 2.

Now consider indistinguishable bosons, shown in the middle chart of Fig. 6.1. Now any two states that differ by an exchange of particles only gets counted once. For example, the classical case of having the • particle in state α and the ○ particle in state β (row 4) is different from the case where the two particles are switched (row 7). But for indistinguishable bosons these are the same state, so it only appears once (row 4 for the bosons).

---

[4] Okay, no one has actually managed to interview a boson to confirm their feelings on the matter, but we can observe their behavior and draw some conclusions.

Classical Particles

	$\|\alpha\rangle$	$\|\beta\rangle$	$\|\gamma\rangle$
1	● ○		
2		● ○	
3			● ○
4	●	○	
5		●	○
6	○		●
7	○	●	
8		○	●
9	●		○

Bosons

	$\|\alpha\rangle$	$\|\beta\rangle$	$\|\gamma\rangle$
1	● ●		
2		● ●	
3			● ●
4	●	●	
5		●	●
6	●		●

Fermions

	$\|\alpha\rangle$	$\|\beta\rangle$	$\|\gamma\rangle$
1	●	●	
2		●	●
3	●		●

Figure 6.1: The distinguishable classical particles are indicated by a filled circle • and an empty circle ∘. The indistinguishable quantum particles are both indicated by a filled circle.

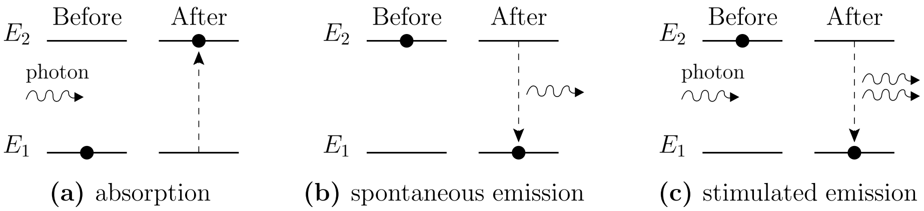
Finally, for fermions it is pretty straightforward. They are indistinguishable, so we just use • for both particles, and we add in the Pauli exclusion principle, which eliminates the cases where both particles are in the same state. This gives the chart on the right in Fig. 6.1.

What do we learn from these charts? Let's consider the case where the three states, $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$, all have the same energy. If these quantum states are in thermal equilibrium, we can appeal to ergodicity to argue that every two-particle microstate is equally probable. And this gives us something interesting.

Look at the probability of finding two particles in the same state. For the distinguishable classical particles, there are 3 cases out of 9 where the particles are in the same state, so it's a probability of 1/3. But for the bosons, there are 3 cases of out 6 where they are in the same state, so a probability of 1/2. Indistinguishability has given the bosons an *enhanced* probability of being in the same state. It is almost as if the bosons have an attractive force pulling them into the same state, but really it's just a property of their statistics.

And this turns out to have quite an impact. When there are a large number of identical bosons in a system, the statistical "attraction" becomes even more powerful and it is sometimes possible to get a macroscopic fraction of the particles to cram together into the same quantum state. This state is called then a Bose condensate. When we achieve this, we end up with a material that exhibits quantum mechanical properties on our everyday macroscopic scale, which is truly bizarre.

We have managed to make Bose condensates with a number of different types of bosons. When we do this with photons, we get a laser. When we



(a) absorption **(b)** spontaneous emission **(c)** stimulated emission

Figure 6.2: Three processes in which photons interact with a two-level atom.

do this with helium-4 atoms, we get a superfluid. When we do this with electrons paired up to be bosons (more below), we get a superconductor. Most recently, physicists have figured out how to get a gas of large atoms, such as rubidium, to form a Bose-Einstein condensate. These are all very different systems with very different properties, and we devote the remainder of the chapter to describing some of them.

6.6 Lasers

The word "laser" is an acronym for **L**ight **A**mplification by **S**timulated **E**mission of **R**adiation. Einstein predicted the essential process of *stimulated emission* (described below) based on his analysis of how light interacts with atoms, but it took nearly a half-century until we were able to construct an operating laser. Now lasers appear in dozens of commercial products including supermarket scanners and blue-ray disc players. They are also used for cutting and shaping in manufacturing, as scalpels in delicate eye surgery, and as tools in countless scientific experiments.

So just what is stimulated emission? Consider a pair of energy levels for an electron in an atom, as shown in Fig. 6.2, with energy difference ΔE. These need not be neighboring energy levels. We have learned already about the processes of absorption and spontaneous emission, which are shown in parts (a) and (b) of Fig. 6.2. Recall that the absorbed or emitted photon must have energy $E_{\text{ph}} = \Delta E$ due to energy conservation.

Einstein realized[5] there must also be a third process of stimulated emission, which involves an incoming photon of energy $E_{\text{ph}} = \Delta E$ and an electron in the upper energy level (see Fig. 6.2c). This appears to be an unlikely candidate for a photon-electron interaction, since the electron is already in the upper energy level. But photons have spin $s = 1$, which makes them bosons, and that means photons are "attracted" to be in the same state. The incoming photon induces the electron to jump down to the lower energy level and emit a second photon with the same energy, in essence "cloning"

---

[5] Einstein used some interesting thermodynamics arguments to discover stimulated emission. You can learn about these in PHYS 222.

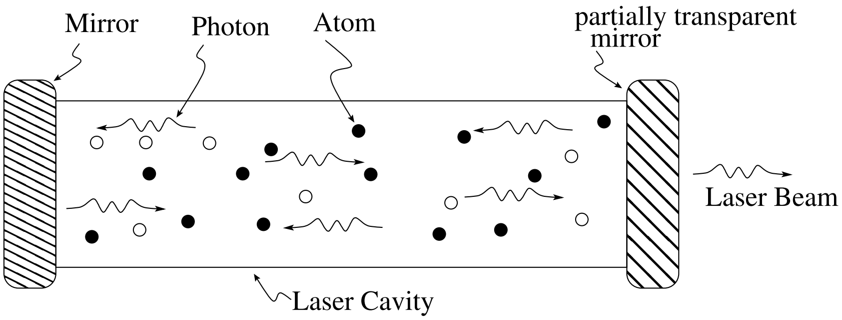


Figure 6.3: Schematic drawing of a gas laser showing laser cavity containing atoms and radiation (photons). Empty circles ○ are atoms in the lower energy states and filled circles ● are atoms in a higher energy state.

itself. In fact, this second photon is in exactly the same quantum state as the incoming photon. This means, among other things, that they have the same momentum vector, and so they are traveling in the same direction.

So how do we make a laser from this? The very first lasers consisted of a gas of atoms contained within a resonant cavity composed of a glass tube sealed at either end with mirrors, as shown in Fig. 6.3. These atoms are chosen to have a suitable pair of energy levels, E_{upper} and E_{lower}, that will be used to create a population of photons with energy $E_{\text{ph}} = E_{\text{upper}} - E_{\text{lower}}$. By putting energy into the system, many of the atoms are placed into the upper energy level. These excited atoms will spontaneously emit photons in random directions. Most of these photons end up simply absorbed by the walls, but the occasional emitted photon will be moving directly horizontal and therefore will stay in the tube, bouncing back and forth between the mirrors. And that means this photon will eventually come near another excited state atom and use stimulated emission to create a second photon. Now these two photons will stay in the tube and, with stimulated emission, each produce more identical photons. Eventually a large population of these horizontally moving photons will build up: that's the *light amplification.*

But there is a villain lurking: every atom that has emitted a photon, either spontaneously or by stimulated emission, is back down at the E_{lower} energy level. These atoms are ready to absorb exactly the photons that we are carefully building up. The only way to win this battle is by numbers. Each stimulated emission event increases the number of photons by one, but each absorption event lowers it by one. So we need to ensure there are more atoms in the upper energy level than in the lower energy level, which is called a *population inversion.*[6]

The technological challenge of making a laser was basically figuring out

---

[6] The reason this is called an *inversion* is because normally a gas at equilibrium at a temperature T has more atoms in the lower state and fewer in the higher energy state.

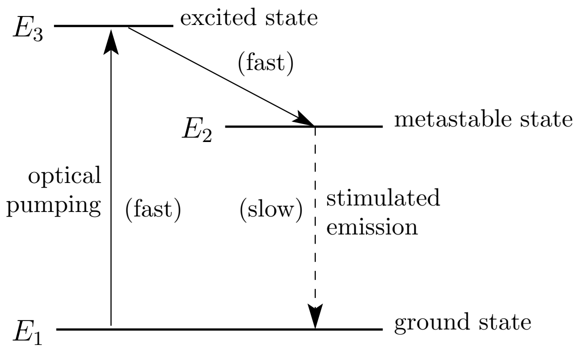


Figure 6.4: Processes in a gas laser. Ground state atoms are quickly excited up to the level E_3 and quickly decay down to the metastable E_2. Thus, there are more atoms at level E_2 than at E_1.

how to achieve the population inversion. The most common way of maintaining a population inversion in the laser is by a method called *optical pumping* in which outside energy is applied to excite atoms from the ground state E_1 to an excited state E_3, as shown in Fig. 6.4. Some excited state atoms decay to the metastable state E_2, which takes a long time to spontaneously decay. As a result, a larger number of atoms end up in the metastable state than in the ground state, which means we have achieved population inversion. Then stimulated emission from E_2 to E_1 is used to create the photons of energies $E_{\text{ph}} = E_2 - E_1$.

What distinguishes laser light from ordinary light? There are three main properties: (1) all the photons have the same energy, or wavelength, so the beam is *monochromatic*; (2) all the photons are traveling in the same direction, so the beam is *collimated*; and (3) all the photons have the same quantum phase, so the beam is *coherent*. While coherence is an interesting property — it allows us to make holograms, for example — we will not be able to discuss it here. The collimation of the beam is what makes lasers good for surgery and for laser pointers: even after traveling some distance, the beam stays narrow with all the intensity contained in a small cross section. Finally, we used the fact that laser light was monochromatic when we did the two-slit interference lab.

6.7 Superfluidity

A superfluid is a fluid (liquid or a gas) with *zero* viscosity. In other words, a flow in a superfluid will *never* slow down. Superfluids are an example of a Bose condensate, and the liquid must be composed of identical molecules that are bosons (with spin $s = 0, 1, 2, \ldots$). Helium-4 is the most common fluid used in studies of superfluidity. It is necessary to cool the system down to a temperature T below a (small) critical value T_c, since molecular motion associated with thermal energy knocks molecules out of the Bose condensate unless the system is very cold.

The lack of viscosity for a superfluid can be understood qualitatively by considering how viscous dissipation works. In a flowing liquid, molecules of the liquid moving with the flow will occasionally strike the side of the container or other liquid molecules and recoil in some other direction. All of the molecules in the fluid undergo these collisions, and the whole fluid eventually slows down to a stop as the energy is gradually converted from motion along the flow into motion in random directions, which is just thermal motion. For a superfluid, however, a significant fraction of the molecules in the system are crammed into the same quantum state, the Bose condensate. An individual molecule in the condensate will not easily exchange energy with the container, since the molecule would have to leave the Bose condensate if it were to move in a direction different than the flow. Superfluidity has not achieved practical status yet because of the fact that the critical temperatures are typically only a couple of Kelvin.

Recently, large atom Bose condensation has been observed experimentally for the first time in gasses. In this context it's called Bose-Einstein condensation, since Einstein predicted this possibility. This only happens at temperatures in the millionths of a Kelvin range, so this was not easy to accomplish. Recent Nobel-Prize winning advances in laser cooling techniques finally enabled physicists to reach these temperatures.

6.8 Superconductivity

Superconductivity, in principle, is quite similar to superfluidity. A superconductor is a material which, if cooled sufficiently, has *zero* electrical resistance. In other words, a superconductor can carry an electrical current without any dissipative losses. Again, though, the material must be cooled below a critical temperature T_c. Superconductivity has been observed in more than 20 metallic elements and in thousands of alloys. The element with the highest critical temperature is niobium, with a critical temperature of 9.3 K.

Recent breakthroughs have led to the discovery of layered copper oxide materials that have a critical temperatures as high at 115 K. While this is still quite cold (-158° C), it is above the boiling temperature for liquid nitrogen, which is easy to make and is relatively inexpensive. Consequently, these *high temperature superconductors*, as they are called, have many practical applications. Here are some examples:

- Zero electrical resistance means that electromagnets can be made that require zero power to maintain. Furthermore, since there is no dissipation, the magnets don't heat up. Once the current is initiated in the magnet, it can continue indefinitely. Magnetic resonance imaging (MRI) devices, in particular, require very strong magnetic fields. Using normal conductors, an MRI device burns up a *lot* of electrical power. Worse, massive refrigeration units are required to cool the electro-magnets (solenoids). Newer MRI devices use wires made of high-temperature superconducting materials, which dramatically lowers the operating cost of the devices.

- Power transmission: without any electrical resistance, superconducting wires could be used to transmit electrical power without any dissipative loss. (Note: this will really only be practical if materials can be found that go superconducting at ambient temperatures.)

- Magnetic levitation: as will be discussed below, superconductors have a property referred to as the *Meissner effect* which enables them to levitate magnets. This principle could potentially be used in the future to make virtually frictionless trains at an economical cost.

- High-speed electronics: superconducting materials make it possible to make circuits with substantially higher switching speeds.

- Magnetic energy storage: a superconducting solenoid with a very large magnetic field can be used like a battery, since the current in the solenoid won't die down.

Superconductors, like superfluids, are based on the principle of Bose condensation, but this time electrons are the particles. Of course electrons are fermions and not bosons. But the electrons pair into composite particles called "Cooper pairs," and these Cooper pairs are then bosons, which can Bose condense. This was not easy to show. After the phenomenon of superconductivity was discovered experimentally, it took almost 50 years for physicists to work out the subtleties of the pairing process and the Bose condensation.

6.8.1 The Meissner Effect

In addition to having zero resistance, superconductors are also noted by a second important property, called the Meissner effect. When a material goes into its superconducting state, any applied magnetic fields are expelled from the material. Put another way, inside a superconducting material the magnetic field is given by $\vec{B} = 0$. This expulsion process is sketched in Fig. 6.5.

The Meissner effect has some interesting implications. First, if $\vec{B} = 0$ inside the superconductor, the superconductor itself must be acting like a magnet that opposes the externally applied magnetic field. The total magnetic field is the sum of the applied field $\vec{B}_{\text{app}}$ and the opposing magnetic

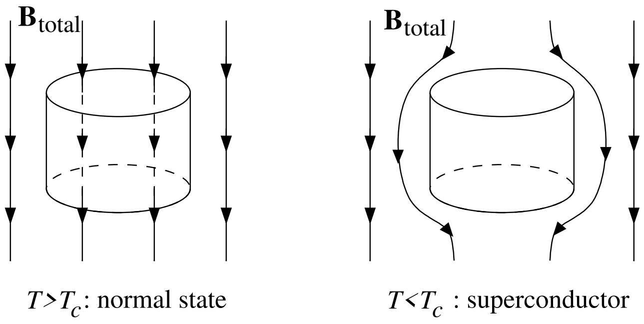


Figure 6.5: The expulsion of the magnetic field in a superconductor, i.e., the Meissner effect.

field from the superconductor $\vec{B}_{\rm sc}$:

$$\vec{B}_{\rm total} = \vec{B}_{\rm app} + \vec{B}_{\rm sc}. \tag{6.9}$$

Inside the superconductor, $\vec{B}_{\rm total} = 0$, so

$$\vec{B}_{\rm sc} = -\vec{B}_{\rm app} \tag{6.10}$$

inside the superconductor. In words, the Meissner effect requires that a superconductor in a magnetic field becomes a magnet that always opposes the applied field.[7] As you will see in the homework assignment, this simple result is the basis behind levitation of magnets over a superconductor (or vice-versa).

The opposing field $\vec{B}_{\rm sc}$ set up by the superconductor requires circulating currents within the superconductor. These currents — referred to as *shielding currents* — flow along the outside of the superconductors. In fact, all of the current passing through a superconductor flows in thin layers near the outside of the superconductor. Because of Ampere's law, if there were any currents within the superconductor, the magnetic field wouldn't be zero inside.

6.8.2 Limits on Superconductivity

Even for temperatures below the critical value ($T < T_c$), the superconductivity can be destroyed by excessive magnetic fields and/or currents. As we increase the strength of an externally applied magnetic field, the magnitude of the shielding currents must also increase in order to provide a

---

[7] Be careful not to confuse this result with Lenz's Law. Lenz's Law says that an induced current in a normal conductor opposes any changes in the magnetic flux. Lenz's Law is inapplicable for superconductors, for which the "induced" (shielding) currents set up a magnetic field to oppose the applied field itself. More about shielding currents in a moment.

shielding field $\vec{B}_{\text{sc}}$ that cancels the externally applied field within the bulk of the superconductor. But there is only a finite amount of energy that is released by the condensation process, so the shielding currents can only be so large. Consequently, there is only so much applied magnetic field that can be shielded. The maximum applied magnetic field is referred to as the *critical magnetic field strength* B_c. If B_{app} exceeds B_c, then the shielding currents aren't strong enough to cancel the magnetic field within the material. As a result, the material remains in its normal, non-superconducting state even though the temperature is below T_c. The critical magnetic field is typically independent of the geometry of the sample.

A phenomenon closely related to the critical magnetic field, and of great practical importance, is the existence of a *critical current.* As an externally applied current (not a circulating shielding current) carried by a superconducting sample is increased, there comes a point at which the superconductor becomes normal, because the current carried by a superconductor produces a field at the surface of the material. If the current is too large, then the magnetic field at the surface can exceed the critical magnetic field B_c.

6.9 Back to the Original Question

At the beginning of the chapter we asked the question whether two identical particles, say two electrons, can be distinguished from one another. Now we can answer the question. Quantum indistinguishability explains both the Pauli exclusion principle and bosonic "attraction." These effects have been seen in countless experiments and phenomena, including the whole field of chemistry. So the answer is no: we cannot distinguish identical quantum particles. It's yet another difference between classical and quantum physics.

Problems

1. In filling up the energy states of an infinite square well potential while obeying the Pauli exclusion principle, it is often stated that one electron can go in spin up and the other electron can go in spin down. This statement is reasonably correct, but it becomes incorrect if you interpret it to mean that the two electrons are in the state $|E_1\!\uparrow \;\; E_1\!\downarrow\rangle$

(a) Explain why this cannot describe the two electrons in the ground state.

(b) Write down the correct state that is intended when two electrons are put into the ground state.

2. Consider the case of an electron and a muon (the electron's cousin, also a spin-1/2 fermion) in a two-particle quantum state.

(a) Is $|e\!\uparrow \;\; \mu\!\downarrow\rangle$ a possible two-particle state? Explain your reasoning.

(b) Is $|e\!\uparrow \;\; \mu\!\uparrow\rangle$ a possible two-particle state? Explain your reasoning.

3. Consider two identical particles that are to be put into two single-particle states labeled by $|\alpha\rangle$ and $|\beta\rangle$.

(a) Make three charts like Fig. 6.1 describing all the possible ways that you can make a two-particle state for classical particles, bosons, and fermions. Use the symbols $\bullet$ and $\circ$ to represent the two distinguishable classical particles, and two $\bullet$'s to represent the indistinguishable quantum particles.

(b) How many states are there for the classical particles? For the bosons? For the fermions?

(c) Assuming that the microstates are equally likely, what is the probability of finding the two particles in the same single-particle state for classical particles? For bosons?

(d) Are your results consistent with the idea that bosons have an enhanced probability (relative to classical particles) of being in the same state?

4. Neutrons have spin $s = 1/2$. Six neutrons are placed in an infinite square well with ground state energy $E_1 = 3\,\text{eV}$. Determine the minimum combined energy these neutrons can have.

5. Explain briefly why a population inversion is necessary for the operation of a laser.

6. The sketches in Fig. 6.6 show the state of a two-level atom and possibly a photon. For each "Before" sketch, make a corresponding "After" sketch and name the process.

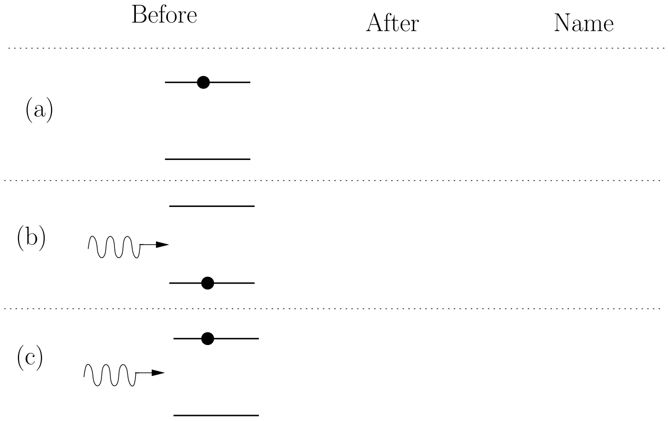


Figure 6.6: Problem 6

7. Superconductors. (Do in Problem Session) Here we investigate some magnetic properties of superconductors.

(a) Closely observe the little cube hovering over the disk. Comment on what you observe. What evidence do you have that this is a superconductor? Can you make the cube spin?

(b) Explain how the superconductor can levitate the magnet.

8. Why would you not expect helium-3 to act as a superfluid at low temperatures, even though helium-4 does?

9. Sketch the magnetic field lines outside the north-pole end of a bar magnet (the field points generally away from the north pole). Now, using the right-hand rule that relates magnetic field to current, sketch the shielding currents that must flow in a flat superconducting plate when the north-pole end of the bar magnet is held just above the plate. Pay particular attention to the direction of the shielding current. Does the current in the plate repel or attract the bar magnet?

10. *Superconducting magnets.* One of the most productive uses of superconductors is in the fabrication of strong electromagnets (e.g., for magnetic resonance imaging in hospitals, or in massive particle accelerators). Consider a superconducting magnet constructed from a solenoid of 1000 turns of superconducting wire with radius 50 cm and length 1 m. A current I is applied, resulting in a magnetic field of 10 Tesla. Once the magnetic field is produced, what power would have to be provided to the electromagnet to maintain the magnetic field?

11. What would happen if dice were indistinguishable bosons? Of course, real dice are too large for quantum effects to be significant, so they are classical, distinguishable objects. As a result, rolling a pair of classical dice yields 36 possible results (1-1, 1-2, 1-3, ..., 6-5, 6-6).

(a) Calculate the probability of rolling doubles with real (classical) dice.

(b) If dice were indistinguishable bosons, then rolling a 2-5 combination would be exactly the same as rolling a 5-2 combination. How many different results are possible for bosonic dice?

(c) How many different ways of rolling doubles are possible with bosonic dice?

(d) Use your results from (a) and (b) to calculate the probability of rolling doubles with bosonic dice. Is the probability higher, lower, or the same as the classical result?

12. Is it possible to create a quantum state in which one electron in an infinite square-well potential is in the $n = 2$ state with spin up and another electron is in the ground state ($n = 1$) with spin up? If so, write it out and make sure that it is antisymmetric. If not, explain why not.

13. Consider *three* identical particles that are to be put into two single-particle states labeled by $|\alpha\rangle$ and $|\beta\rangle$.

(a) Make three charts like Fig. 6.1 describing all the possible ways that you can make a three-particle state for classical particles, bosons, and fermions. Use the symbols $\bullet$, $\circ$ and $\times$ to represent the three distinguishable classical particles, and three $\bullet$'s to represent the indistinguishable quantum particles.

(b) How many states are there for the classical particles? For the bosons? For the fermions?

(c) Assuming that the microstates are equally likely, what is the probability of finding the three particles all in the same single-particle state for classical particles? For bosons?

(d) Show how your results are consistent with the idea that bosons have an enhanced probability (relative to classical particles) of being in the same state, and consistent with the Pauli exclusion principle.

14. A ruby laser is a solid-state laser that uses a synthetic ruby crystal as the medium in which stimulated emission occurs as opposed to individual atoms in a gas laser. The diagram below shows the energy levels used in the ruby crystal along with their approximate energies relative to the ground state.

(a) What is the frequency of the radiation needed to optically pump the ruby crystal and therefore create the necessary population inversion?

(b) What is the wavelength of the emitted laser beam due to stimulated emission?

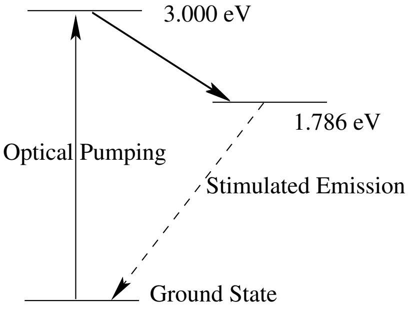


Figure 6.7: Ruby laser energy level diagram for Problem 14

15. A superconducting wire of circular cross section has a radius of 1.5 cm. The critical magnetic field that this particular superconductor can withstand is $B_c = 20\,\mathrm{T}$. Based on this information, calculate the maximum current that this superconducting wire is capable of carrying. *Hint:* use Ampere's law.

16. This problem refers to Fig. 6.1, which shows all the possible ways to put two particles into three states $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$.

(a) For classical particles, take the solid circle to represent particle 1 and the open circle to represent particle 2. Then the two-particle state represented in row 4 would be written $|\alpha\;\beta\rangle$, while the two-particle state in row 7 would be $|\beta\;\alpha\rangle$. Using this notation, write out the classical two-particle states for each of the nine states given.

(b) For bosons, the two-particle states must be symmetric under interchange of particles 1 and 2. Write out the appropriate two-particle states for each of the six states given.

(c) For fermions, the two-particle states must be antisymmetric under interchange of particles 1 and 2. Write out the appropriate two-particle states for each of the three states given.

Chapter 7

Three-Dimensional Wavefunctions and Semiconductors

7.1 Introduction

The basics of quantum mechanics and its interpretation can be understood by first looking at one-dimensional problems. This is what we have done so far in this unit. But the real world has three spatial dimensions, and most real systems have more than one particle to keep track of. The extension of the Schrödinger equation to three dimensions is straight-forward, but the math can get a little complicated.

We will confine our study primarily to the case of electrons in atoms. A key feature of wavefunctions in three dimensions (3D) is that the allowed quantum states are described by **three** quantum numbers instead of one. For the states of electrons in atoms, each quantum number has a distinct physical interpretation, as we will see.

Also in this chapter we describe the basics of semiconductor physics, which follows from the discussion of energy bands in Chapter 4. We will explain how the conductivity of materials can be understood in terms of these bands, and we'll discuss how these quantum properties have led to some really cool (and tremendously important) techniques that form the basis of modern electronics.

7.2 Quantum in 3D

In earlier chapters we have seen how to use the Schrödinger equation to solve for the wavefunctions of a particle in one dimension. In three dimensions, the Schrödinger equation can be written in spherical coordinates (r, θ, ϕ) as:

$$-\frac{\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+U\psi=E\psi. \tag{7.1}$$

At this point, you might be hyperventilating, but don't worry — we will restrict ourselves to cases where everything is spherically-symmetric (with no θ- or ϕ-dependence), so most of the terms in this equation drop out. And we won't ask you to *solve* this equation, but will instead use the approach from earlier where we give you a solution and ask for you to test to see if it does or does not satisfy the equation.

Example 1. Schrödinger equation for 3D Harmonic Oscillator

Show that the test wavefunction $\psi(r) = Ae^{-br^2}$ is a solution to Eq. (7.1) for the harmonic oscillator with potential energy function $U = \frac{1}{2}kr^2$. If it is a solution (which it is), determine the energy of the quantum state.

Solution: If you have never seen a partial derivative such as $\partial\psi/\partial r$, fear not — it is the same thing as a regular derivative where everything else (in this case, θ and ϕ) is treated as constant. And since the test wavefunction for this example doesn't have any θ or ϕ's in it, we can quickly simplify Schrödinger's equation by dropping out any terms that have derivatives with respect to θ or ϕ. So, we now have:

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{2}kr^2\psi=E\psi. \tag{7.2}$$

The derivative $\partial\psi/\partial r = Ae^{-br^2}(-2br)$ and the expression $\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) = -6br^2Ae^{-br^2} + 4b^2r^4Ae^{-br^2}$. So, Eq. (7.2) becomes:

$$-\frac{\hbar^2}{2mr^2}\left[-6br^2Ae^{-br^2}+4b^2r^4Ae^{-br^2}\right]+\frac{1}{2}kr^2\psi=E\psi. \tag{7.3}$$

The terms Ae^{-br^2} are present in each term, so they cancel. Simplifying,

$$\frac{3b\hbar^2}{m}-\frac{2b^2\hbar^2}{m}r^2+\frac{1}{2}kr^2=E \tag{7.4}$$

$$\rightarrow r^2\left(\frac{1}{2}k-\frac{2b^2\hbar^2}{m}\right)+\left(\frac{3b\hbar^2}{m}-E\right)=0 \tag{7.5}$$

and this is satisfied only if both the constant and r^2 terms separately add to zero. Setting the r^2 terms to zero gives us $b = \frac{\sqrt{mk}}{2\hbar}$ and $E = \frac{3b\hbar^2}{m} = \frac{3\hbar^2}{m}\frac{\sqrt{mk}}{2\hbar} = \frac{3}{2}\hbar\sqrt{k/m}$.

So, **yes**, the test function $\psi(r) = Ae^{-br^2}$ satisfies Schrödinger's equation for a harmonic oscillator as long as b and E are given by the values above.

A particularly important case for the 3D Schrödinger equation (Eq. (7.1)) is that of a hydrogen atom where the potential energy function $U = -ke^2/r$. The complete general solutions for this equation, describing all possible states for an electron in a hydrogen atom, are beyond the scope of this course.[1] We'll be content with checking a few of the simplest cases (you'll be doing this for homework, using an approach very similar to the example above). But certain general observations will be made.

The first observation is that the spatial wavefunction $\psi(x, y, z)$ for any single-particle, 3D quantum system involves three quantum numbers. For atoms, each of the three quantum numbers (n, l, and m_l) which arise in solving this particular 3D problem can be associated with a particular spherical coordinate. The principal quantum number n is associated with the radial coordinate r, and is related to the number of nodes in the wavefunction as you move outward from the origin on a radial line. The orbital quantum number l is associated with the polar angle θ (measured from the "North pole"). When $l = 0$, the associated wavefunction has no dependence on θ. The orbital magnetic quantum number m_l is associated with the azimuthal angle ϕ; when $m_l = 0$, there is no ϕ-dependence.[2]

Actually, there is a fourth quantum number m_s associated with the spin angular momentum of the electron — this is the same quantum number that we discussed in the previous chapter. So, a full description of the state of an electron in an atom requires the specification of four quantum numbers: $|n, l, m_l, m_s\rangle$.

The second observation is that the quantum numbers are directly connected to measurable physical quantities of the atom. For solutions of Eq. (7.1), the energy of the electron in hydrogen is determined to a high degree of accuracy by the principal quantum number n.[3] The energy formula for an electron in a state of quantum number n is

$$E_n = -\frac{1}{n^2}\frac{mk^2e^4}{2\hbar^2} = -\frac{13.6\,\text{eV}}{n^2}. \tag{7.6}$$

---

[1] But not beyond the scope of a class in physical chemistry or upper-level quantum mechanics — something to look forward to if you are a chemistry or physics major!

[2] In other math classes you have taken you may have used ϕ for the polar angle and θ for the azimuthal angle. This is just a matter of convention. In this course we will consistently use the definitions given in the text above.

[3] For real atoms, there are slight modifications to the energy based on the other quantum numbers, but we won't discuss those slight corrections in this course.

Another useful physical quantity is the electron's orbital angular momentum $\vec{L}$, whose magnitude is given in terms of the orbital quantum number l as

$$|\vec{L}| = \sqrt{l(l+1)}\hbar. \tag{7.7}$$

The z-component of the orbital angular momentum L_z is given by

$$L_z = m_l\hbar, \tag{7.8}$$

and the z-component of the spin angular moment S_z is given by

$$S_z = m_s\hbar. \tag{7.9}$$

7.3 The Hydrogen Atom

We are now ready to talk about the actual wavefunctions of the simplest atom: hydrogen. Specifically, we consider the wavefunctions for the electron that "orbits" around the single proton in the nucleus. We put the word "orbits" in quotes because the behavior of the electron is quite different from the behavior of a satellite orbiting around the Earth. Because of the small mass of an electron and the small size of an atom, quantum wave-like effects are quite significant.

Because of the Uncertainty Principle, we can't specify both the position and the velocity of the electron. An electron orbiting in an atom is typically smeared out into a probability wave around the atom, so there really *isn't* a "position" per se. And because of the fact that the motion of the electron (whatever *motion* really means in the quantum world) isn't in a straight line, it isn't meaningful to specify the velocity or momentum of the electron. Instead we solve the Schrödinger equation (7.1) to find wavefunctions representing various states of the hydrogen atom. As usual, the square of the magnitude of the wavefunction is to be interpreted as a probability density, in this case, probability per unit ***volume***.

By solving Schrödinger's equation (7.1), each hydrogen wavefunction solution is characterized by three quantum numbers (n, l, m_l), and is written as $\psi_{nlm}(r, \theta, \phi)$. (The quantum number m_s isn't needed to specify the wavefunction.) For instance, the lowest energy (ground) state wavefunction of hydrogen $(n = 1, l = 0, m_l = 0)$ is calculated to be

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}}\; e^{-r/a_0}. \tag{7.10}$$

The excited state $(n = 2, l = 1, m_l = 0)$ is given by

$$\psi_{210} = \frac{1}{\sqrt{32\pi a_0^3}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \cos\theta \tag{7.11}$$

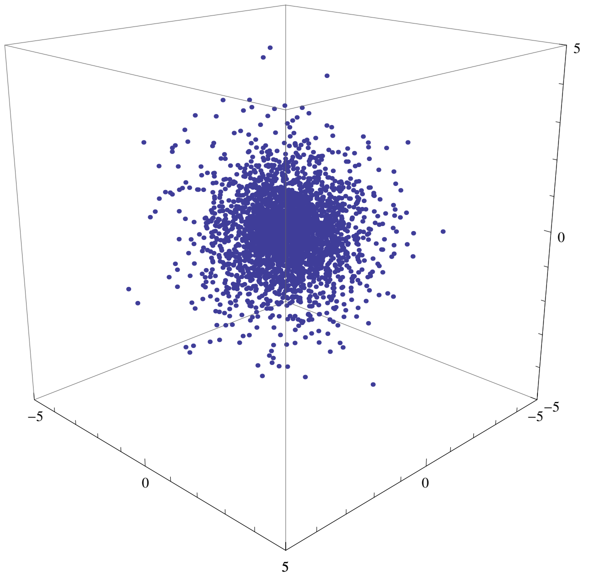


Figure 7.1: Plot of probability density for an electron in state ψ_{100} of hydrogen.

and the excited states $(n = 2, l = 1, m_l = +1)$ and $(n = 2, l = 1, m_l = -1)$ are

$$\psi_{211} = -\frac{1}{\sqrt{64\pi a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\sin\theta\, e^{i\phi}. \tag{7.12}$$

$$\psi_{21-1} = \frac{1}{\sqrt{64\pi a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\sin\theta\, e^{-i\phi}, \tag{7.13}$$

respectively. We can use these wavefunctions to calculate their probability densities $|\psi_{nlm_l}|^2$ and then plot these probability densities to get a visualization of the "location" of the electron in this state. Figure 7.1 shows such a plot for the ground state ψ_{100}. In this plot, the probability density for finding the electron at any location is related to the density of dots in the plot. Thus, the probability density is greatest at the center $r = 0$ and drops off exponentially as you move away from the origin, in agreement with the wavefunction in Eq.(7.10). Also we notice that the probability density is spherically symmetric, implying that it doesn't depend on the direction in space, which follows from Eq. (7.10) since there is no dependence on the coordinates θ and ϕ.

In the case of the excited states ($n = 2$, $l = 1$, $m_l = 1$, 0,-1), it is actually real linear combinations of the wavefunctions given in equations (7.12) and (7.13) that make up the familiar atomic p-orbitals that are important in chemical bonding. For instance, the linear combination

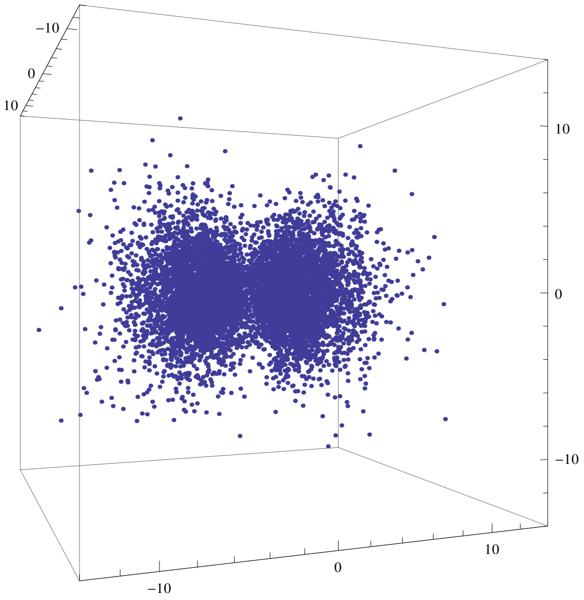


Figure 7.2: Plot of probability density for an electron in an orbital state ψ_{2p_x} of hydrogen.

$$\begin{aligned}\psi_{2p_x} &= -\frac{1}{\sqrt{2}}\left[\psi_{211} - \psi_{21-1}\right] \\ &= \frac{1}{\sqrt{32\pi a_0^3}}\left(\frac{r}{a_0}\right) e^{-r/2a_0}\sin\theta\cos\phi\end{aligned} \tag{7.14}$$

corresponds to what is known as the "p_x" orbital that you might have seen in chemistry textbooks. The probability density is shown in Fig. 7.2. In this case you will notice that the probability density is not spherically symmetric as in the ground state, but instead has two lobes that extend outward along the x-axis.

Finally, here are a few rules relating the atomic quantum numbers n, l, m_l. We won't derive them here in PHYS 212. You'll have to take PHYS 222 for more details about this.

- The possible values for the principal quantum number n are

$$n = 1,\ 2,\ 3,\ \ldots \tag{7.15}$$

- Given a principle quantum number n for an electron, the orbital quantum number l can have the following values

$$l = 0,\ 1,\ \ldots,\ n-1. \tag{7.16}$$

  So, for example, if $n = 1$, then l can only be 0. If $n = 2$, then l can be 0 or 1. If $n = 3$, then l can be 0, 1, or 2. Etc.

- Given an orbital quantum number l, the magnetic quantum number m_l can have the following values

$$m_l = -l,\ -(l-1),\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ l-1,\ l. \tag{7.17}$$

  So, for example, if $l = 0$, then m_l can only be 0. If $l = 1$, then m_l can be -1, 0, or 1. If $l = 2$, then m_l can be -2, -1, 0, 1, 2, etc.

The rule for the quantum number m_s is the same as was discussed in Chapter 5 for an electron: $m_s = -1/2$ or $+1/2$, regardless of the other three quantum numbers n, l and m_l.

Some of the consequences of these rules will be explored in the problems.

7.4 Periodic Table

We now have the basic tools to explain the periodic table of the elements. Electrons are fermions since they have a half-integral spin quantum number ($s = 1/2$ for an electron). Consequently, electrons obey the Pauli Exclusion Principle (see Section 6.4): no two electrons can be in the same state. So, if there is more than one electron in an atom, no two of them can have the same combination of quantum numbers n, l, m_l and m_s.

Consider lithium, which has three protons in the nucleus and three electrons orbiting it. The first two electrons can go into the ground state, but Pauli demands that the third electron go into a new state. If the lithium atom is in its lowest possible energy state, that third electron must be in some $n = 2$ orbital. Most of the chemical behavior of atoms is due to the shape and size of their outer electron orbitals, so the Pauli exclusion principle is what makes lithium act very differently than helium.

This process repeats many times over to make up the periodic table. The more electrons there are, the higher the energy levels must be filled, similar to the five particle system in Example 6-1. For some specific orbital, meaning some specific values of n, ℓ and m_ℓ, there are two possible single-electron states: $|n\,\ell\,m_\ell\uparrow\rangle$ and $|n\,\ell\,m_\ell\downarrow\rangle$, and we can put two electrons into the antisymmetric combination of these two states. Any additional electrons will have to go into a new orbital.

Were it not for the Pauli Exclusion Principle (i.e., if it didn't apply to electrons in atoms), then all elements would have all of their electrons in the lowest energy state with $n = 1$ (except when excited). Were that the case, then there would be no chemical diversity — each element would behave almost identically to each other, and there would be no chemical interactions. Chemistry would be really dull in this case. Even worse, we wouldn't be around to notice that chemistry had become dull.

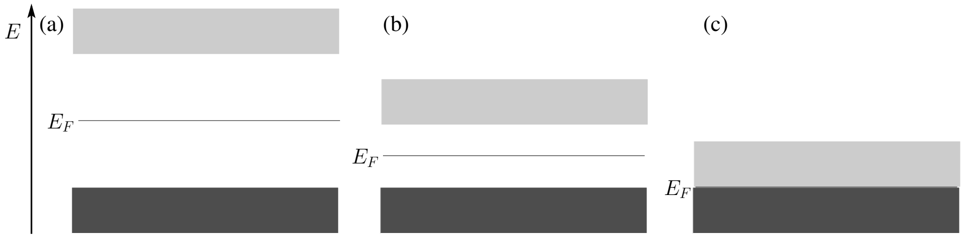


Figure 7.3: Highest valence and lowest conduction bands (with Fermi Level) for: (a) an insulator; (b) a semiconductor; and (c) a conductor. The dark shading denotes filled energy levels.

7.5 Conductors, Semiconductors, and Insulators

For the second half of this chapter, we return to our discussion from Chapter 4 of quantized energy levels and band structure. It turns out that some clever manipulations of these materials can result in some really important electronic components. The material in this chapter is the basis behind some of the most important developments in modern electronics.

We return to the discussion of energy bands from Chapter 4. First, we need to discuss the electrical conductivity of a material and how it relates to the band structure. Figure 7.3 shows the highest energy, filled valence band and the lowest energy, unfilled conduction band for a typical material. We define an energy level referred to as the "Fermi energy" E_F that denotes the boundary between the highest energy filled level and the lowest unfilled level for a material at a temperature of absolute zero ($T = 0$ K). By convention, for a situation like that in Fig. 7.3, E_F is in the middle of the forbidden energy gap.

The electrical properties of a material depend critically on the band gap energy E_g. For the case in Fig. 7.3a, the band gap energy is so large that electrons in the filled valence band rarely jump up to the conduction band. Since the valence band is completely full, there is no possibility for electrical conduction if an electric field is applied to this material, because there is no way for any electron in the valence band to change its energy since all the other energy levels in the valence band are filled. (Recall: Pauli's Exclusion Principle applies here, since electrons are fermions.) And there are no electrons in the conduction band either. So, Fig. 7.3a corresponds to an electrical insulator.

An electrical conductor is shown in Fig. 7.3c. In this case, the band gap energy $E_g = 0$, so if an electrical field is applied, electrons in the valence band can easily be excited into one of the available energy levels in the conduction band.

Figure 7.3b corresponds to a "semiconductor." The energy band struc-

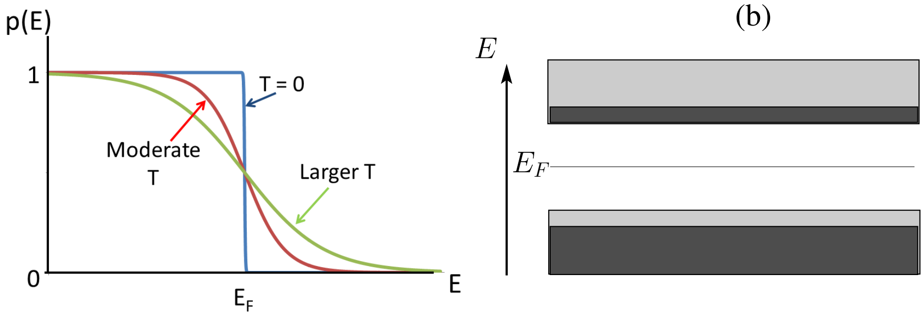


Figure 7.4: (a) Fermi-Dirac relation (Eq. (7.18)) for three temperatures. (b) Energy band diagram for a temperature large enough to excite several electrons from the valence to conduction bands.

ture is similar to the insulator of Fig. 7.3a, except that the band gap energy E_g is smaller. This is important because the smaller but non-zero E_g makes it possible to "turn on" and "turn off" electrical conductivity, a **very** useful property.

We discuss two ways in which a semiconductor can be manipulated to produce good conductivity.

7.5.1 Temperature Dependence of Conductivity for a Semiconductor

If the temperature of the material is not zero, then there is thermal energy that can cause electrons in the valence band to become excited into the conduction band. The larger the temperature, the larger the interatomic collision energies that can excite electrons. The temperature dependence is captured well by the *Fermi-Dirac* probability distribution function:

$$p(E) = \frac{1}{1 + e^{(E-E_F)/k_B T}}. \tag{7.18}$$

This relation is plotted in Fig. 7.4a for three different temperatures. For a material at absolute zero ($T = 0$), all of the energy levels for $E < E_F$ are filled (with probability $p(E) = 1$), whereas all energy levels for $E > E_F$ are empty ($p(E) = 0$). If the temperature is not zero, though, thermal excitations result in a decrease in p for $E < E_F$ and an increase for $E > E_F$.

If the temperature is large enough, then the probability $p(E)$ for electrons to be found for energies in the conduction band may become non-negligible, and there may be sufficient electrons excited into the conduction band for the material to become a good conductor. An important question: **how large is "large enough?"** The answer to this question can be seen by noticing the factor of k_BT in Eq. (7.18). This factor should look familiar to you from our thermodynamics unit from PHYS 211 — this is the well-known *Boltzmann factor* that gives a characteristic energy for thermal collisions. If the temperatue in a semiconductor is such that k_BT is comparable to or greater than the band gap energy E_g, then there will be sufficient thermal energy for the semiconductor to be a good conductor. This is shown conceptually in Fig. 7.4b: for a sufficiently large temperature, some of the higher energy levels in the valence band become empty and some of the lower energy bands in the conduction band contain electrons. This enables conduction not only for electrons in the conduction band but also for electrons in the valence band who now have open energy levels if an electric field is applied.

Example 2. Probabilities from Fermi-Dirac.

Lead sulfide (PbS) has a band gap energy of 0.37 eV. (a) Use an approximation where $k_BT \approx E_g$ to determine a temperature for which you would expect PbS to be an excellent conductor. (b) Calculate the probability of finding an electron with energy at the bottom of the conduction band for temperatures of 400K and 4000K.

Solution: (a) As a reasonable approximation, if the temperature is such that $k_BT \approx E_g$, then there should be sufficient energy from thermal collisions to excite electrons from the valence to the conduction band. So, $T \approx E_g/k_B = (0.37\,\text{eV})/(8.62 \times 10^{-5}\,\text{eV}\cdot\text{K}^{-1}) = 4290\,\text{K}$.

(b) We can calculate probabilities from the Fermi-Dirac equation (Eq. (7.18)). Since the Fermi energy E_F is in the center of the band gap, the lowest conduction band energy $E = E_F + E_g/2$, so for $T = 400\,\text{K}$,

$$\begin{aligned} p(E) &= \frac{1}{1+e^{(E-E_F)/k_BT}} = \frac{1}{1+e^{E_g/2k_BT}} \\ &= \frac{1}{1+e^{(0.37\,\text{eV}\cdot\text{K}^{-1})/(2\times 8.62\times 10^{-5}\,\text{eV}\cdot\text{K}^{-1}\cdot 400\,\text{K})}} \\ &= 0.0047 \end{aligned} \tag{7.19}$$

This temperature is about 10 times lower than the value that we calculated in part (a), so it makes sense that the probability is small for finding an electron at the bottom level of the conduction band.

If you repeat the calculation for $T = 4000\,\text{K}$, you'll find a probability of 0.37, which is quite appreciable. This is consistent with the result in part (a) which predicts a temperature of 4290 K for appreciable conduction.

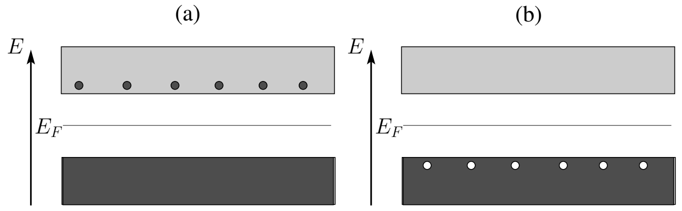


Figure 7.5: (a) n-type semiconductor; donor atom impurities are added to provide some extra electrons which occupy some of the levels in the conduction band. (b) p-type semiconductor; acceptor atom impurities are added that remove electrons from the valence band, leaving conducting "holes" behind.

In the previous example, it is tempting to say that if $p(E) = 0.0047$ for an electron to be found in the lowest conduction-band energy state, then the material will be an insulator. But remember how many electrons are in a typical solid (a **lot**!), so even if the probabilities are low, there will still be some electrons that can conduct electricity. But with higher and higher temperatures there would be more and more conduction band electrons.

So, semiconductors conduct electricity better at higher temperatures. That can be both a good thing and a bad thing. On the good side, this principle can be used to make very sensitive and precise temperature measurement devices called *thermistors*. On the bad side, the electronic devices that we'll discuss in the next section can fail if the semiconductor gets too hot. This is why it is so critical for electronic devices to be cooled.

7.5.2 Doping and n- and p-Type Semiconductors

There is another way that you can make a semiconductor conduct: doping, i.e., add impurities to a semiconductor with a different number of valence electrons than the primary semiconductor material. For example, silicon is a common "Group IV" semiconductor element with 4 valence electrons; i.e., 4 electrons in the highest energy (unfilled) shell. If a silicon semiconductor is doped with trace amounts of an impurity "Group V" element such as arsenic with 5 valence electrons, 4 of the valence electrons in the impurity will fill the valence band along with the silicon atoms, but there will be one additional outer-shell electron for each impurity atom. Arsenic is considered to be a "donor impurity" for silicon semiconductors since it donates an additional electron. The additional donated electrons are mobile charge carriers in the conduction band, as shown in Fig. 7.5a.[4] Semiconductors doped with donor atoms are called "n-type" semiconductors, because they have mobile charge carriers that have negative charge (electrons).

---

[4] This is a simplification: the impurities alter the band structure, but the basic principle is the same.

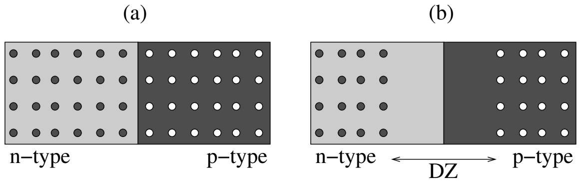


Figure 7.6: (a) A pn junction (in the absence of diffusion or a depletion zone) made by connecting an n-type semiconductor on the left (with negative mobile charges — black dots) and a p-type semiconductor on the right (with positive mobile charges — white dots). (b) A more realistic pn junction; n-type electrons and p-type holes diffuse across the junction and annihilate each other, resulting in a depletion zone (DZ) around the junction with no mobile charge carriers.

Doping a semiconductor with "acceptor impurities" results in "p-type" semiconductors, as shown in Fig. 7.5b. An example of a p-type semiconductor would be silicon (Group IV) doped with impurities from Group III (with 3 valence electrons), such as boron. Since the impurity atoms have only 3 valence electrons (unlike the 4 for the bulk silicon atoms), each impurity atom would fill only 3 out of the 4 available energy levels, leaving an unfilled "hole" in the valence band. This hole provides other electrons in the valence band with an opportunity to change their energies in response to an applied electric field. Consequently, the presence of holes in the valence band allows for conductivity. An applied electric field results in electrons in the valence band moving opposite the field into available holes; the result is that the holes move in the direction of the electric field. Conceptually, the holes themselves act like positive, mobile charge carriers; hence the term *p-type* to describe these semiconductors.

Things become very interesting if a p-type semiconductor is joined with an n-type semiconductor. Figure 7.6a shows an n-type semiconductor (on the left) with mobile electrons (dark circles) connected to a p-type semiconductor (on the right) with mobile holes (white circles). But in the absence of an applied electric field, the mobile electrons and holes diffuse around the material, moving randomly in the sample, including moving across the p-n junction. Invariably, mobile electrons and holes near the junction come into contact with each other and *annihilate*, with the mobile electron dropping into the open energy level left by the hole. This annihilation process removes both the electron and the hole from the conduction process. The result is that in the vicinity of the junction between the n- and p-type materials, there is a *depletion zone* (DZ) with no mobile charge carriers, as shown in Fig. 7.6b. The annihilation process at the junction produces a *bias* electric field that eventually stops the diffusion of the electrons and holes across the junction, stabilizing the width of the DZ.

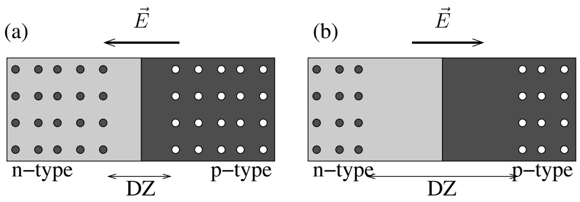


Figure 7.7: (a) Forward-biased pn junction with an electric field pointing from the p-type toward the n-type semiconductor. This electric field pulls n-type charges (electrons) to the right and p-type charges (holes) to the left, decreasing the size of the depletion zone. (b) Reverse-biased pn junction with an enlarged depletion zone.

7.6 Diodes, Photodiodes, and LEDs

A p-n junction like that shown in Fig. 7.6 forms what is know in electronics as a *diode* — a device that enables current to pass only in one direction. Diodes are used for *rectifiers* which turn AC (alternating current) power from the outlets in your house into DC (direct current) power needed for most devices. Diodes are used to protect sensitive electronic devices from accidental reverse voltages. And diodes form some of the fundamental elements in digital computer circuitry.[5]

To understand how a p-n junction works as a diode, consider Figs. 7.6 and 7.7. The key is the depletion zone (DZ) at the junction. Since there are no mobile charge carriers in the DZ, then that part of the device is a good electrical insulator. For a p-n junction to conduct, the DZ has to be eliminated.

If you want to run an electrical current through a p-n junction, you need to apply an electric field. If the field points from the p-type to the n-type semiconductor (as in Fig. 7.7a), the holes in the p-type material are pulled in the direction of the electric field toward the junction, and the electrons in the n-type material are pulled in the direction opposite the electric field, also toward the junction. Consequently, the DZ shrinks. If the field is strong enough[6] the holes and electrons meet at the junction, and the DZ is completely gone; consequently, the entire device conducts and the current can pass. So, a strong enough electric field in this *forward-biased* direction will eliminate the DZ and produce an electrical current.

On the other hand, if an electric field is applied pointing from the n-type to the p-type semiconductor, the field pushes the holes and mobile electrons *away* from the junction, making the DZ even larger. With the DZ intact, the

---

[5] For a full appreciation of the importance of diodes — and other electronic elements — consider PHYS 235 (“Applied Electronics”).

[6] It turns out that for silicon semiconductors, there is a “turn-on voltage” of 0.6 V that needs to be exceeded for a diode to conduct in the forward-biased direction.

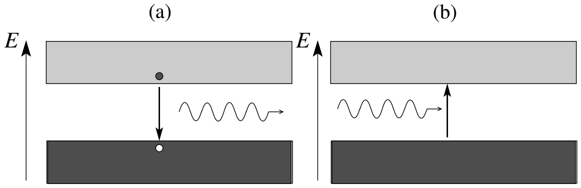


Figure 7.8: (a) Photon emitted when an n-type charge carrier (an electron) annihilates a p-type charge carrier (a hole); conceptually, the electron drops into the open, lower-energy state represented by the hole. (b) Excitation of a valence electron by an incoming photon. The absorption of the photon results in the formation of an n- and p- charge carrier pair.

p-n junction won't conduct electricity in this *reverse-biased* direction. So, a p-n junction works as a diode — it allows current to flow in one direction but not the other.[7]

But more can be done with a diode (p-n junction) than just passing current only in one direction. When a forward-biased diode is conducting a current, mobile electrons and holes are continually annihilating each other at the junction. Each time an electron-hole pair annihilates, a photon is emitted, as shown in Fig. 7.8a. This is the principle behind what are called *light-emitting diodes* (LEDs), which can be found in almost any piece of modern electronics. The little red and blue indicator lights on your cell phone are LEDs. The "flash" that your cell phone uses to take pictures at night (which you probably use more as a flashlight when walking back to your dorm at night) is an LED. Most scoreboards at games are made with LEDs. And you can now buy LED light bulbs which are *significantly* more efficient and reliable than standard, incandescent light bulbs.

The process works in reverse. If a photon with suitable wavelength and energy hits a p-n junction (Fig. 7.8b), it can excite an electron from the valence to the conduction band, producing an electron-hole pair in the DZ. Some residual electric fields in the DZ[8] sweep this electron and hole away from the DZ and produce a small electrical current that can be measured. As a result, a diode can be used as a light detector — it is referred to as a *photodiode* when used this way. This is the basic idea behind the detectors used in modern digital cameras, including the camera in your cell phone.

Finally, an introduction to semiconductor physics would not be complete with a discussion of the *transistor*, the development of which completely revolutionized modern technology. A basic transistor can be made out of a sandwich of a p-type semiconductor between two pieces of n-type semi-

---

[7] It isn't an ideal diode because of the turn-on voltage, but there are ways of getting around this limitation.

[8] We won't explain the origin of this residual electric field here.

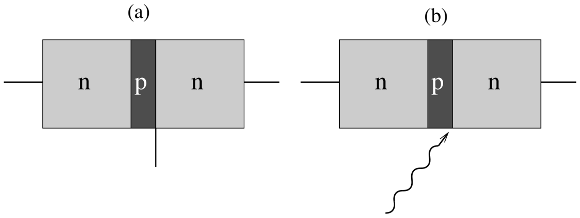


Figure 7.9: (a) An npn transistor. (b) Phototransistor.

conductors (Fig. 7.9). For your homework problems, you will explain (by considering the depletion zones at each of the two p-n junctions) how a transistor works. The result is a device that will not pass an electrical current in *either* direction unless a very small electrical current is provided at one of the p-n junctions. That very small, second current effectively "turns on" or "turns off" the larger current flowing lengthwise through the device. As you will also see in your homework, the current can also be turned on by light shining on the p-n junction, as shown in Fig. 7.9b for a device referred to as a phototransistor.

The use of a transistor as an electronic "on-off switch" has tremendous implications for modern technology. Without electronic switches, most of modern electronic technology would be useless. Simple examples of the use of transistor switches abound. A transistor switch turns on the blue indicator light on your phone when you have a new email or the red indicator light if your phone is charging. A transistor switch turns on and off the display of your phone or the LED light that you use to walk home at night. In a computer, billions of transistor switches are rapidly turning on and off small currents that enable the computer to perform its tasks.

Transistors are also very important for power amplification. As an example, your cell phone detects electromagnetic waves transmitted by a nearby cell tower, but the detected signal is very weak. A transistor amplifier makes the signal stronger so that your cell phone can decode the signal and produce a time-varying voltage which is then amplified by another transistor circuit to produce a sound loud enough for your ear to hear.

The npn transistor was the first truly electronic switch that didn't depend on vacuum-tube technology.[9] There are many advantages to transistors over vacuum tubes, the most significant being that they can be miniaturized and printed into integrated circuits. Current computer CPUs have over a **billion** transistors. Think about that: that is a **lot** of transistors.

---

[9] Vacuum tubes were common in all radios and TVs up until the 1950s, but you will be hard-pressed to find them now.

Problems

1. The so-named Paschen series of radiation emitted by hydrogen corrersponds to transitions from states of principle quantum numbers $n = 4$, 5, 6, …, to the state $n = 3$.

   (a) Calculate the frequencies and wavelengths of the three lowest energy photons emitted from hydrogen in the Paschen series.

   (b) In what region of the electromagnetic spectrum are these photons?

2. An electron in a hydrogen atom is in a state of principle quantum number $n = 4$. Using the notation (n, l, m_l, m_s) to specify a state, write all of the possible states for this electron. [HINT: There should be a total of 32 states.]

3. Find the maximum possible magnitude for the orbital angular momentum $|\vec{L}|$ of an electron in the state of principle quantum number $n = 7$ of a hydrogen atom.

4. The orbital angular momentum of the electron in a hydrogen atom has a magnitude $|\vec{L}| = 2.585 \times 10^{-34}\,\text{J}\cdot\text{s}$. What is the minimum possible energy for this electron in the hydrogen atom?

5. Determine the principle quantum number n and orbital quantum number l for a hydrogen atom whose electron has energy $-0.850\,\text{eV}$ and orbital angular momentum $|\vec{L}| = \sqrt{12}\hbar$.

6. Show that the probability that a state with energy ΔE above the Fermi energy will be occupied is equivalent to the probability that a state with energy ΔE *below* the Fermi energy will be *unoccupied.* Explain in 1 or 2 sentences why this result makes sense intuitively, when considering how conduction works in a semiconductor.

7. A one-dimensional periodic potential composed of 10 side-by-side finite square well potentials has a valence band with 10 energy levels from a low of 0.81 eV to a high of 1.10 eV and a conduction band with 10 levels ranging from 3.19 eV to 4.39 eV . The Fermi energy (in the middle of the band gap) is 2.145 eV for this system. Assume that there are 20 electrons in this system (this is analogous to 10 atoms with 2 valence electrons each).

(a) Calculate the probability that an electron will be found in the energy state $E_{11} = 3.19\,\text{eV}$ (i.e., in the lowest conduction band state) for $T = 300\,\text{K}$(room temperature), $3,000\,\text{K}$, and $30,000\,\text{K}$.

(b) Calculate the values of $k_B T$ for each of the temperatures from part (a). Do the probabilities that you calculated in (a) make sense, given these values for $k_B T$?

(c) Now, download the Excel sheet "FDForTenWells.xlsx" from the calendar page. This worksheet calculates Fermi-Dirac probabilities for electrons to occupy any of the 10 conduction band energies. The sheet also calculates an expected average number of electrons in each of these levels (by multiplying the probability by 2, since there are two energy levels per state) and adds up the total number of electrons expected in the conduction band. Use this worksheet to calculate the estimated number of conduction band electrons for $T = 300$, 3000, and 30000 K. For each of these temperatures, would you expect this system to be an electrical insulator or electrical conductor?

(d) if there were only 10 electrons (e.g., 10 atoms with only 1 valence electron instead of 2), how would that affect the conduction properties of this system?

8. Lead sulfide (PbS) has a band gap energy of 0.37 eV.

(a) Calculate the value of $k_B T$ for temperatures 30 K, 300 K (room temperature), 3,000 K and 30,000 K. At these temperatures would you expect PbS to be a poor, moderate or good electrical conductor?

(b) Use the Fermi-Dirac relation to calculate the probability of the lowest conduction band energy state being occupied for temperatures 30, 300, 3,000 and 30,000 K. Are these results consistent with your answers from (a)? (Note: it's not necessary for *every* conduction band level to be filled for a material to be a reasonable conductor, but the probabilities should at least be on the order of 10^{-5} or better.)

9. Some recent studies have investigated the possibility of using DNA strands as fundamental building blocks of nanotechnology. One possibility is the use of DNA for nanoelectronic devices. Experimental studies of a particular DNA sequence (Poly(dA)*Poly(dT) with a B-type structure[10]) measured a band gap of 2.7 eV between valence and conduction bands. Would you expect this DNA strand to be a good electrical conductor at room temperature? Explain. (A "k_BT" analysis is sufficient — you don't need to use Fermi-Dirac distributions here.) If it is a good conductor, approximate how low a temperature would be needed to make it a poor conductor. If it is a poor conductor, approximate how large a temperature would be needed to make it a good conductor.

10. A silicon photodiode has a band gap energy of 1.17 eV.

(a) Calculate the maximum wavelength of electromagnetic radiation that this photodiode can detect.

(b) Why can't a silicon photodiode detect EM radiation with wavelengths larger than the value calculated in part (a)?

(c) Give an argument as to how a silicon photodiode *can* detect EM radiation with wavelengths *smaller* than the value calculated in part (a).

11. Calculate the wavelength (and state the color — or "IR" or "UV" if infrared or ultraviolet) of electromagnetic radiation emitted from a light-emitting diode (LED) made from the following materials:

(a) Gallium arsenide (band gap 1.43 eV);

(b) Gallium (III) Phosphide (band gap 2.26 eV);

(c) Gallium nitride (band gap 3.4 eV).

12. White light LED bulbs can be made by using blue or violet LEDs that are coated with phosphor (fluorescent) coatings. (a) Explain in a few sentences how this kind of approach can produce light that appears white. (b) Why would it *not* be possible to make a white light LED with a red LED as the source? (Note: white-light LED bulbs can also be made with coatings of quantum dots or can be made with combinations of red, green and blue LEDs.)

---

[10] "DNA Electronics," M. Taniguchi and T. Kawai, Physica E **33**, 1-12 (2006).

13. (a) Draw a sketch of a p-n junction in the absence of any imposed electric fields. In the sketch, show the depletion zone (DZ) and sketch mobile carriers within the p- and n-type semiconductors away from the DZ. Make sure that you have the correct type of charge carriers (+ or -) in the correct material.

(b) Now, make a similar sketch, but for the case with an applied electric field pointing from the n- to the p-type semiconductor. Is the DZ larger or smaller in this case than for the case with $E = 0$ from part (a) Explain why. Do you expect this p-n junction to conduct electricity readily in the direction of this electric field? Explain.

(c) Repeat part (b), but for an electric field pointing from the p- to the n-type semiconductor, answering the same questions.

(d) Does this device act like a diode; i.e., a device that passes current readily in one direction but not in the other? In what ways does a p-n junction deviate from an "ideal" diode?

14. Transistors. Figure 7.9(a) is a sketch of a sandwich made of 2 n-type semiconductors with a thin p-type layer in between.

(a) A wire is connected to each of the ends of this npn sandwich. Give a short explanation as to why this npn device would not be expected to conduct electricity in either direction. (Hint: consider what happens to the DZ at each junction if you apply an electric field in either direction.)

(b) A third wire is connected to the right pn junction. If a very small current is fed into the device through this third wire, there can be a significant conduction of electricity between the two end wires if the current is going to the left. Explain why. (Hint: consider the depletion zone and what happens if you inject some electrons or holes into it.)

(c) Instead of a third wire, you can make a phototransistor that turns on and off the current between the ends by shining light on the p-type layer. Explain why light can turn on the current. (Hint: this will only work if the light has a wavelength smaller than a particular critical value.)

15. The wavefunction for the ground state of the hydrogen atom ($n = 1, l = 0, m_l = 0$) is

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0},$$

where a_0 is a constant which can be written in terms of fundamental constants as $a_0 = \frac{\hbar^2}{m_e k e^2}$.

(a) Show that this wavefunction is a solution to the 3D Schrödinger equation

$$-\frac{\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{ke^2}{r}\psi = E\psi$$

(Hint: since there are no θ or ϕ terms in the wavefunction, the partial derivatives $\partial\psi/\partial\theta$ and $\partial\psi/\partial\phi$ are both zero. That will simplify things a **lot**.)

(b) Determine the value of E required for ψ_{100} to be a solution. Compare your result with Eq. (7.6).

16. The ground state wavefunction of the electron in the hydrogen atom is spherically symmetric which means that the wavefunction $\psi(r)$ can be written solely in terms of the radial coordinate r representing the distance between the proton and electron.

(a) What does the quantity $|\psi(r)|^2$ mean physically?

(b) Show that the volume of a thin spherical shell of radius r and thickness dr is $4\pi r^2\, dr$. (You can use the approximation for small dr that the volume is the surface area of the sphere times dr.)

(c) In spherical coordinates, the ground state solution of the Schrödinger equation for the hydrogen atom is

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0},$$

where a_0 is the same constant as from the previous problem. Use the result of part (b) to write an expression for the probability that the electron is in a spherical shell of radius r and thickness dr.

(d) Calculate the radius of the shell (of constant thickness dr) where the electron is most likely to be found.

Chapter 8

Quantum Entanglement

8.1 Introduction

In our final chapter on quantum mechanics we introduce the concept of *entanglement*. This is a feature of two-particle states (or multi-particle states) in which the probabilities of the particles are linked in ways that cannot be described by classical physics. Entanglement is at the heart of current research in quantum cryptography, quantum computing, and quantum teleportation. As we shall see, it also leads to some extremely counter-intuitive behavior. We can construct entangled states in which particle 1 is a large distance away from particle 2, and yet a measurement on particle 1 can instantly influence particle 2. Einstein called this “spooky action at a distance.”

Finally, we will be able address the question of *completeness* of quantum states. When we say a spin has a probability of being measured in the spin up state, is this probability just a reflection of our ignorance — the spin really has some value and we just don’t know it — or is the particle’s spin fundamentally not determined until we make the measurement? For decades physicists assumed that this was an unanswerable question, but in recent years we have found ways to turn this into an experimental test. To learn what the experiments have found, read on!

8.2 Entangled Two-Particle States

For a single electron, any possible spin state can be written as

$$|\psi\rangle = c_+\,|{+z}\rangle + c_-\,|{-z}\rangle = c_+\,|{\uparrow}\rangle + c_-\,|{\downarrow}\rangle, \tag{8.1}$$

with the appropriate choice of the complex numbers c_+ and c_-. For notational simplicity we will switch to writing the spin states in terms of up and down arrows for the remainder of this chapter.

Now consider a two-particle state made up of two different kinds of particle. To be specific, let’s take an electron and a positron, which is

state	electron	positron
$\lvert\uparrow\uparrow\rangle = \lvert\uparrow\rangle\lvert\uparrow\rangle$	$S_z = +\hbar/2$	$S_z = +\hbar/2$
$\lvert\uparrow\downarrow\rangle = \lvert\uparrow\rangle\lvert\downarrow\rangle$	$S_z = +\hbar/2$	$S_z = -\hbar/2$
$\lvert\downarrow\uparrow\rangle = \lvert\downarrow\rangle\lvert\uparrow\rangle$	$S_z = -\hbar/2$	$S_z = +\hbar/2$
$\lvert\downarrow\downarrow\rangle = \lvert\downarrow\rangle\lvert\downarrow\rangle$	$S_z = -\hbar/2$	$S_z = -\hbar/2$

Table 8.1: The four possible states with definite S_z values.

a particle with the same mass and spin as the electron, but it has a $+e$ charge.[1] What is the expression analogous to Eq. (8.1) for any possible two-particle state?

There are four possible states that have definite values for the z-component of the spin, listed in Table 8.1. These four states make a *basis* in the sense that any possible two-particle state for the electron and positron is a superposition of these states:

$$|\psi\rangle = c_1 \lvert\uparrow\uparrow\rangle + c_2 \lvert\uparrow\downarrow\rangle + c_3 \lvert\downarrow\uparrow\rangle + c_4 \lvert\downarrow\downarrow\rangle. \tag{8.2}$$

If the two particles were both electrons, we would need to build in indistinguishability by choosing coefficients c_1 to c_4 that make an anti-symmetric state. But that's not the case here: particle 1 is an electron and particle 2 is a positron, so they are distinguishable from each other. The state vector does not need to be anti-symmetric.

We have also introduced in Table 8.1 the notation that we can write a two-particle state with definite values of S_z in a factored form: $\lvert\uparrow\downarrow\rangle = \lvert\uparrow\rangle\lvert\downarrow\rangle$. This is simply two different ways to write the same thing: the electron has spin up and the positron has spin down. Both notations have their advantage and we'll switch back and forth between them. But be careful when using the "factored" states: the electron state vector must always be on the left and the positron state vector on the right. In other words, $\lvert\uparrow\rangle\lvert\downarrow\rangle \neq \lvert\downarrow\rangle\lvert\uparrow\rangle$.

Now let's examine the behavior of this two particle state with an example. Consider an electron-positron pair in the state

$$|\psi\rangle = 0.9\lvert\uparrow\uparrow\rangle + 0.1\lvert\uparrow\downarrow\rangle + 0.3\lvert\downarrow\uparrow\rangle + 0.3\lvert\downarrow\downarrow\rangle. \tag{8.3}$$

First, let's compute the probabilities associated with a measurement of the positron's S_z spin component.

There are two ways that the positron could be found to have $S_z = +\hbar/2$, as both the $\lvert\uparrow\uparrow\rangle$ state and the $\lvert\downarrow\uparrow\rangle$ state have the positron spin up. So we need to add together the probability of measuring $\lvert\uparrow\uparrow\rangle$ and the probability

---

[1] You may have heard of positrons at least indirectly: they are used in Positron Emission Tomography, also known as a PET scan.

of measuring $|{\downarrow\uparrow}\rangle$:

$$\begin{aligned}\text{Prob(positron spin up)} &= \text{Prob}(\,|{\uparrow\uparrow}\rangle) + \text{Prob}(\,|{\downarrow\uparrow}\rangle) \\ &= |0.9|^2 + |0.3|^2 \\ &= 0.81 + 0.09 = \boxed{0.90.} \end{aligned} \tag{8.4}$$

Note that we did not add the coefficients first and then square them. That would have led to the $|0.9 + 0.3|^2 = 1.44$, which is a nonsensical result for a probability!

Similarly, we calculate the probability of the positron being found with spin down by combining the probabilities of the $|{\uparrow\downarrow}\rangle$ and $|{\downarrow\downarrow}\rangle$ states:

$$\text{Prob(positron spin down)} = |0.3|^2 + |0.1|^2 = \boxed{0.10.} \tag{8.5}$$

Notice that we have a normalized state and the total probability of the positron having either spin up or spin down adds to one: $0.90 + 0.10 = 1$.

Now let's pose a question that gets at the central topic of this chapter:

> Suppose we start out again with the state $|\psi\rangle$ and first measure the electron's S_z value. Will this measurement affect the state of the positron?

Here's a way to approach it. Let's write the basis states in their factored form, and see if we can combine terms:

$$\begin{aligned}|\psi\rangle &= 0.9\,|{\uparrow}\rangle\,|{\uparrow}\rangle + 0.1\,|{\uparrow}\rangle\,|{\downarrow}\rangle + 0.3\,|{\downarrow}\rangle\,|{\uparrow}\rangle + 0.3\,|{\downarrow}\rangle\,|{\downarrow}\rangle \\ &= \;|{\uparrow}\rangle\Big(0.9\,|{\uparrow}\rangle + 0.1\,|{\downarrow}\rangle\Big) + \;|{\downarrow}\rangle\Big(0.3\,|{\uparrow}\rangle + 0.3\,|{\downarrow}\rangle\Big). \end{aligned} \tag{8.6}$$

Look at what we have done here: we have used a distributive rule to factor our two-particle state. This is allowed, but keep in mind that in any product we must keep the electron state vector on the left and the positron state vector on the right.

To make this expression more useful, we would like the terms in parentheses to be normalized states. We can achieve this by multiplying and dividing by the appropriate factor, something like

$$|\psi\rangle = c_+\,|{\uparrow}\rangle\left(\frac{0.9}{c_+}\,|{\uparrow}\rangle + \frac{0.1}{c_+}\,|{\downarrow}\rangle\right) + c_-\,|{\downarrow}\rangle\left(\frac{0.3}{c_-}\,|{\uparrow}\rangle + \frac{0.3}{c_-}\,|{\downarrow}\rangle\right). \tag{8.7}$$

As you will check in the homework, the terms in parentheses will be normalized states as long as $c_+ = \sqrt{0.82}$ and $c_- = \sqrt{0.18}$. Putting in those values gives

$$|\psi\rangle = \sqrt{0.82}\,|{\uparrow}\rangle\,|\phi_1\rangle + \sqrt{0.18}\,|{\downarrow}\rangle\,|\phi_2\rangle \tag{8.8}$$

with the normalized *positron* states

$$\begin{aligned} |\phi_1\rangle &= \frac{9}{\sqrt{82}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{82}}\,|{\downarrow}\rangle \\ |\phi_2\rangle &= \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle. \end{aligned} \tag{8.9}$$

Now, $|\psi\rangle$ written in the form Eq. (8.8) is nothing other than the $|\psi\rangle$ we started with in Eq. (8.3). But it is now much more useful: we can finally address our central question (does an electron spin measurement affect the positron state) using tools we've already learned for states.

An electron S_z measurement will have a probability of 0.82 of finding the electron spin up. If this occurs, the state will collapse to the new state

$$|\psi_{\text{new}}\rangle = |{\uparrow}\rangle\,|\phi_1\rangle. \tag{8.10}$$

Alternately, there is a probability 0.18 of the electron being found to have spin down, in which case the state collapses to

$$|\psi_{\text{new}}\rangle = |{\downarrow}\rangle\,|\phi_2\rangle. \tag{8.11}$$

We can now determine the probabilities for the subsequent positron spin measurement by looking at the coefficients in the normalized states $|\phi_1\rangle$ and $|\phi_2\rangle$. And here is what we find:

- If we do not measure the electron's spin, the positron has a probability 9/10 of being spin up.
- If we measure the electron's spin and find it to be spin up, then the positron has a probability 81/82 of being spin up.
- If our electron spin measurement had instead found it to be spin down, then the positron has a probability 1/2 of being spin up.

This is the essence of entanglement! Measurements on one particle affect the state of the other particle.

8.3 Separable States

Not all two particle states are entangled. An example of a state that is not entangled would be the following:

$$|\psi\rangle = \frac{1}{2}\,|{\uparrow\uparrow}\rangle + \frac{1}{2}\,|{\uparrow\downarrow}\rangle + \frac{1}{2}\,|{\downarrow\uparrow}\rangle + \frac{1}{2}\,|{\downarrow\downarrow}\rangle. \tag{8.12}$$

Following the procedure of the previous section leads to

$$\begin{aligned} |\psi\rangle &= |{\uparrow}\rangle\left(\frac{1}{2}\,|{\uparrow}\rangle + \frac{1}{2}\,|{\downarrow}\rangle\right) + |{\downarrow}\rangle\left(\frac{1}{2}\,|{\uparrow}\rangle + \frac{1}{2}\,|{\downarrow}\rangle\right) \\ &= \frac{1}{\sqrt{2}}\,|{\uparrow}\rangle\left(\frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\right) + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\left(\frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\right) \end{aligned} \tag{8.13}$$

where in the second line we have made normalized positron states inside the parentheses.

And now something interesting has happened: the two positron states inside parentheses are identical. This means that if we measure the electron's spin, the resulting state collapse leads to the same positron state, regardless of what we obtain for the electron spin. The positron is no longer entangled with the electron!

Mathematically, we note that having the same positron state in both parentheses above means we can factor our state even further:

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\right)\left(\frac{1}{\sqrt{2}}\,|{\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow}\rangle\right) \tag{8.14}$$

In this form we can see directly that the two-particle state is really just a product of a single-particle electron state and a single-particle positron state. In general, whenever this happens that we can fully factor the two-particle state into

$$|\psi\rangle = |\phi_{\text{electron}}\rangle\,|\phi_{\text{positron}}\rangle. \tag{8.15}$$

then the particles are not entangled. We call these *separable states*.

In summary, two-particle states are separable if they can be fully factored into a product of single-particle states. And if they cannot be factored, then they are entangled.

8.4 Bell States

Not all entangled states are equally entangled. Some are very nearly separable: imagine tweaking the coefficients of 1/2 in Eq. (8.12) to values like 0.49 and 0.51. Other states are far from separable.

There exist two-particle states that are, in some sense, maximally entangled. In these states, which are called Bell states,[2] the S_z value of the positron is completely determined by the S_z measurement on the electron. One example is the following:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|{\uparrow\uparrow}\rangle + \frac{1}{\sqrt{2}}\,|{\downarrow\downarrow}\rangle. \tag{8.16}$$

For an electron-positron pair in this state, an electron spin measurement is equally likely to give spin up or spin down. But a subsequent positron measurement will *always* find the positron to have the same spin that you just measured for the electron.

By the way, the entanglement goes in both directions: you could measure the positron spin first and collapse the state of the electron to be the same spin as the positron. Also, it is possible to construct Bell states where the electron and positron spins are exactly opposite.

---

[2] Named after John Bell, an Irish physicist. You'll be encountering more of his handiwork shortly.

8.5 EPR Paradox

You may have heard at some point that Einstein never accepted quantum mechanics. There is a grain of truth in it: Einstein did not dispute the predictions of quantum theory, but he never believed that quantum mechanics was a *complete* theory. For example, he did not accept the idea that a particle does not have a definite position. He was convinced that there must be some more information available that we simply haven't understood yet, and that if we could work out this additional aspect to the theory then particles would once again have precise positions, as in classical physics. This additional information, which is not in the quantum theory but Einstein felt must be a part of reality, is called a *hidden variable.*

In trying to prove his point, Einstein, along with Podolsky and Rosen, formulated a thought experiment that leads to an apparent paradox, called the EPR paradox. Specifically, consider creating an electron-positron pair in the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|{\uparrow\downarrow}\rangle - \frac{1}{\sqrt{2}}\,|{\downarrow\uparrow}\rangle. \tag{8.17}$$

This is fairly easy to do: our particle accelerators create many electron-positron pairs in their collision events, and conservation of angular momentum requires that the pair have a total angular momentum of zero. Hence, they pop into existence in precisely this Bell state.

Now imagine that you have physically separated the electron-positron pair without disturbing their spin states. Maybe, by chance, you get a pair heading directly away from each other, and you keep them from interacting with any other particles until they are a full meter away from each other.

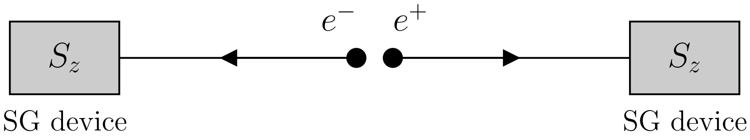


Figure 8.1: The EPR thought experiment. An electron-positron pair is created in state $|\psi\rangle$ given in Eq. (8.17). The particles are physically separated and then their spins are measured by Stern-Gerlach devices.

After they are separated, you send the electron through a Stern-Gerlach device to measure its S_z spin component, as shown in Fig. 8.1. The result is random, with an equal chance of spin up or spin down. But as soon as you have made that measurement, the positron spin is no longer random. Essentially, you have collapsed the state of both particles in that instant, even though they are separated by a meter.

If that doesn't seem strange, imagine first letting the particles get a kilometer apart and then doing the measurement. Or even a light year apart. Quantum theory says the distance is irrelevant: if you can get the particles entangled and keep them entangled, then collapse of state due to a measurement "travels" from one particle to the other instantly, regardless of their separation. This is what Einstein referred to as "spooky action at a distance." And he felt that this argument proved that quantum mechanics was not a complete theory.

It is worth emphasizing again that Einstein did not dispute the result predicted by quantum theory. But he thought a more natural explanation was that the state $|\psi\rangle$ in Eq. (8.17) is not really a complete description of the electron-positron pair. Rather, there is some more information, the *hidden variable*, which if understood would cause these measurements to be not mysterious at all. It helps to have a specific example of a hidden variable theory in mind to digest this.

Example 1. A Hidden Variable Theory

Imagine that, contrary to what you learned about quantum mechanical spin in Chapter 5, an electron really does have a definite spin vector $\vec{S}$ with all three components precisely determined. None of this probability stuff. So this $\vec{S}$ vector is the hidden variable.

We can reconcile this idea with the Stern-Gerlach experiment by claiming that we don't completely understand spin measurement. When we measure S_z, we somehow end up getting the value $+\hbar/2$ any time that $\vec{S}$ has a positive z-component. And we end up measuring the value $-\hbar/2$ any time $\vec{S}$ has a negative z-component.

This simple example of a hidden variable theory explains the EPR experimental results in a straightforward way. The electron-positron pair are created with net angular momentum of zero, so we must have $\vec{S}^{\text{elec}} = -\vec{S}^{\text{pos}}$. These $\vec{S}$ vectors are established at the beginning, when the two particles are in contact. And while we don't know the value of either $\vec{S}^{\text{elec}}$ or $\vec{S}^{\text{pos}}$, we do know their z-components have opposite sign. So the EPR experiment will always give opposite signs for the electron and positron S_z components, but not due to any communication at a distance.

This is a pretty compelling argument! But interestingly, the EPR paradox did not sway most physicists to abandon quantum mechanics. One reason was that Bohr showed that this instantaneous state collapse across a large distance did not actually violate special relativity. No actual object travels from the electron to the positron. But more significantly, no actual information travels either: the person measuring the electron cannot send a signal via the state collapse ("spin up if by land, spin down if by sea")

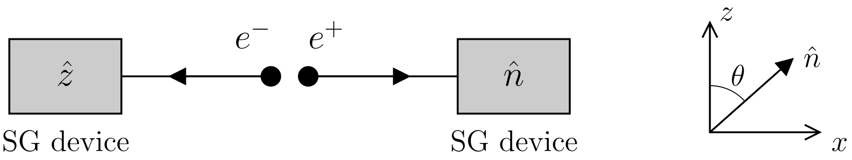


Figure 8.2: The Bell thought experiment, which is the same as the EPR experiment except that the positron Stern-Gerlach detector is rotated.

because they cannot control what they will find for the electron spin. They just get a sequence of random results, and the positron measurer also gets a sequence of random results. Comparison later reveals that the random results are connected: every time the electron observer measures S_z^{elec} to be spin up, the positron observer measures S_z^{pos} to be spin down and visa versa.

The main reason that the EPR paradox did not push physicists to abandon quantum mechanics, though, is that it didn't seem to matter. After all, the hidden variable theories predicted the same results as quantum mechanics. Since physicists knew how to use quantum mechanics and it kept giving successful predictions about the microscopic world, there was little motivation to worry about whether we should replace quantum mechanics with a hidden variable theory.

8.6 Bell's Proposed Experiment

Bell changed all that. To everyone's great surprise, in 1964 he constructed a thought experiment where quantum mechanics and hidden variable theories actually predict different results. This thought experiment suddenly made it an experimental question, so nature could tell us who was right!

Bell's clever idea rested on a variation of the EPR thought experiment where the Stern-Gerlach device that is measuring the positron spin is rotated. The electron's spin is still measured the same way, so we will find $S_z = +\hbar/2$ or $-\hbar/2$ as before. But the positron's spin is measured along some direction $\hat{n}$ that makes an angle θ with respect to the z axis, as shown in Fig. 8.2. For simplicity, we will take $\theta = 45°$.

8.6.1 QM Prediction for the Bell Experiment

If we rotate our positron detector to be 45° from the z-axis and measure this component of the spin, call it S_{45°, what possible values will we find? Just like with any spin component, we'll find $+\hbar/2$ and $-\hbar/2$. And after the measurement, the spin will be in either the $|\nearrow\rangle$ state or the $|\swarrow\rangle$ state, just as an S_x measurement results in either the $|+x\rangle$ or $|-x\rangle$ states.

We can relate our familiar up and down spins to this new pair of states:

$$|\uparrow\rangle = b_+ \, |\nearrow\rangle + b_- \, |\swarrow\rangle \qquad\qquad |\downarrow\rangle = b_- \, |\nearrow\rangle - b_+ \, |\swarrow\rangle. \tag{8.18}$$

The coefficients, b_+ and b_- are given by

$$b_+ = \sqrt{\frac{1+\cos\theta}{2}} \qquad\qquad b_- = \sqrt{\frac{1-\cos\theta}{2}}, \tag{8.19}$$

where θ is measured from the z-axis as shown in Fig. 8.2.

We'll create an electron-positron pair in the initial state given by Eq. (8.17). Let's explore what quantum mechanics predicts for the Bell Experiment. We're going to consider the case where an observer measures S_z^{elec} to be spin down. According to quantum mechanics, a measurement of $S_z^{\text{elec}} = -\hbar/2$ collapses the state of our electron-positron pair to be $|\psi\rangle = |{\downarrow\uparrow}\rangle$. To put it another way, if we measure the state of the electron to be $|\downarrow\rangle$, quantum entanglement says that the state of the positron must be $|\uparrow\rangle$. The subsequent positron measurement will find S_{45° to be either $+\hbar/2$ or $-\hbar/2$ with probabilities that depend on the b_+ and b_- variables. Since the positron state is $|\uparrow\rangle$, the probability of measuring $S_{45^\circ}^{\text{pos}} = +\hbar/2$ is $|b_+|^2$ and the probability of measuring $S_{45^\circ}^{\text{pos}} = -\hbar/2$ is $|b_-|^2$. In your homework, you will calculate these probabilities.

8.6.2 Hidden Variable Prediction for the Bell Experiment

Now here is where it gets interesting: Bell constructed a proof that *any* hidden variable theory[3] with the detectors oriented as in Fig. 8.2 and with $\theta = 45^\circ$ will result in probabilities that are incompatible with the predictions of quantum mechanics. So, either the hidden variable theory or quantum mechanics must be wrong!

While we won't provide a proof of Bell's theorem in full generality here, we will show how to determine the predictions of hidden variable theory introduced in Example 1 of Section 8.5. According to this hidden variable theory, the electron and positron have fully determined spin values $\vec{S}^{\text{elec}} = -\vec{S}^{\text{pos}}$, but we just don't know them.

Picture the positron spin variable $\vec{S}^{\text{pos}}$. Since we don't know its value and we aren't in any way controlling how it is set when the electron-positron pair are created, it is reasonable to assume it is equally likely to be pointing in any direction. We can represent all possible directions of $\vec{S}^{\text{pos}}$ by the surface of a sphere, where the vector $\vec{S}$ has its tail on the origin. A circular cross section of this sphere is shown in Fig. 8.3. The dashed line in the figure is the boundary between the positive and negative values for $S_{45^\circ}^{\text{pos}}$. That is, for spins $\vec{S}^{\text{pos}}$ above and to the right of the dashed line a measurement would

---

[3] Technically, any *local* hidden variable theory, to satisfy the lawyers and mathematicians.

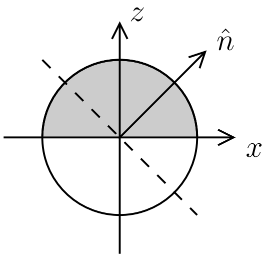


Figure 8.3: The circle represents all possible $\vec{S}^{\text{pos}}$ directions. After a measurement of $S_z^{\text{elec}} = -\hbar/2$, the possible directions of $\vec{S}^{\text{pos}}$ are limited to the upper hemisphere (shaded region).

result in $S_{45^\circ}^{\text{pos}} = +\hbar/2$. The region below and to the left corresponds to $S_{45^\circ}^{\text{pos}} = -\hbar/2$.

Now consider the case where we have measured the electron's z-component of spin and found $S_z^{\text{elec}} = -\hbar/2$. This happens half the time, and when it happens the positron $\vec{S}^{\text{pos}}$ must be somewhere in the upper hemisphere, shown as the shaded region in Fig. 8.3. This agrees with the QM prediction so far, where if we measure $S_z^{\text{elec}} = -\hbar/2$, we know that we would also measure $S_z^{\text{pos}} = +\hbar/2$. But, here, rather than the positron actually *being in a spin up state*, the hidden variable theory says that the original $\vec{S}^{\text{pos}}$ should determine the probabilities for measuring $S_{45^\circ}^{\text{pos}}$ as spin up or down. According to the figure, a fraction 3/4 of the shaded region results in a value $S_{45^\circ}^{\text{pos}} = +\hbar/2$. Bell constructed a proof that for *any* hidden variable theory, this fraction of 3/4 is the *maximum* probability of measuring $S_{45^\circ}^{\text{pos}} = +\hbar/2$.

So, let's recap. We create an electron-positron pair in the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle. \tag{8.20}$$

We make a measurement of the z-component of the electron's spin, and we find that the electron is spin down. Quantum mechanics says that the positron is thus in the state $|\uparrow\rangle$, which means the probability of measuring $S_{45^\circ}^{\text{pos}} = +\hbar/2$ is $|b_+|^2$, or 85%. The hidden variable theory says that the positron spin is determined by its original $\vec{S}^{\text{pos}}$, and we have a $\leq$75% probability of measuring $S_{45^\circ}^{\text{pos}} = +\hbar/2$.

. . . so which is right?

8.7 The Aspect Experiment

These experiments are quite hard to do. In fact, the best experimental tests of Bell's inequality to date is done not with electrons and positrons, but rather with polarization states of photons. The first definitive test was performed in the 1980s in France by the research group of Aspect, and their data came down firmly on the side of . . . drum roll . . . quantum mechanics![4]

Perhaps you have been holding out hope that the spin of an electron could have some precise value and that quantum mechanics was just an incomplete theory, but Bell and Aspect have ruled out that possibility! It appears that quantum mechanics is a complete theory, and quantum mechanics states really do represent the most knowledge possible about the condition of a particle. The randomness of uncertainty in quantum mechanics that we've been studying in this unit *is real.*

---

[4] Aspect and two other physicists were awarded the Nobel Prize for this work in 2022.

Problems

1. An electron-positron pair is in the state

$$|\psi\rangle = \frac{1}{\sqrt{10}}\,|\uparrow\uparrow\rangle + \frac{1}{\sqrt{5}}\,|\uparrow\downarrow\rangle + \frac{1}{\sqrt{5}}\,|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}\,|\downarrow\downarrow\rangle.$$

(a) Calculate the probability that a measurement of the electron S_z spin component will give the value $+\hbar/2$.

(b) For the same state $|\psi\rangle$, calculate the probability that a measurement of the positron S_z component will give the value $-\hbar/2$.

2. Show that the state $|\psi\rangle$ in Eq. (8.7) takes the form shown in Eq. (8.8), with positron states $|\phi_1\rangle$ and $|\phi_2\rangle$ given by Eq. (8.9).

3. An electron-positron pair is in the state

$$|\psi\rangle = \frac{1}{\sqrt{10}}\,|\uparrow\uparrow\rangle + \frac{1}{\sqrt{5}}\,|\uparrow\downarrow\rangle + \frac{1}{\sqrt{5}}\,|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}\,|\downarrow\downarrow\rangle.$$

(a) Convert this state to the form $|\psi\rangle = c_+\,|\uparrow\rangle\,|\phi_1\rangle + c_-\,|\downarrow\rangle\,|\phi_2\rangle$. Determine the coefficients c_+ and c_- and the normalized positron states $|\phi_1\rangle$ and $|\phi_2\rangle$.

(b) Check that your coefficients satisfy $|c_+|^2 + |c_-|^2 = 1$.

(c) Is your answer consistent with Problem 1(a)?

4. An electron-positron pair is in the state

$$|\psi\rangle = 0.6\,|\uparrow\rangle\,|\phi_1\rangle + 0.8\,|\downarrow\rangle\,|\phi_2\rangle$$

with positron states

$$|\phi_1\rangle = \sqrt{\frac{1}{3}}\,|\uparrow\rangle - \sqrt{\frac{2}{3}}\,|\downarrow\rangle \qquad |\phi_2\rangle = \sqrt{\frac{2}{3}}\,|\uparrow\rangle + \sqrt{\frac{1}{3}}\,|\downarrow\rangle.$$

(a) What is the probability that a measurement of the z-component of the **positron's** spin will find a result $S_z^{\mathrm{pos}} = +\hbar/2$?

(b) You now measure the z-component of the **electron's** spin and find a value $S_z^{\mathrm{elec}} = +\hbar/2$. Write down the new state $|\psi_{\mathrm{new}}\rangle$ of the electron-positron system immediately after this measurement.

(c) What is the probability that a measurement of the z-component of the positron's spin will now find a result of $S_z^{\mathrm{pos}} = +\hbar/2$? Is it the same as your answer to part (a)?

(d) Is the original state $|\psi\rangle$ an entangled state? How can you tell?

5. Construct a separable state that has all four coefficients in Eq. (8.2) not equal to zero.

6. Two Bell states were provided in the reading, in Eqs. (8.16) and (8.17). Find two more Bell states.

7. Let's verify the details for the quantum mechanics prediction for the Bell experiment with the positron detector rotated 45° from the $+z$ direction. We'll use a simulation of the Stern-Gerlach Experiment.

   (a) Go to `https://phet.colorado.edu/sims/stern-gerlach/stern-gerlach_en.html`

   (b) In the simulation rotate the angle to 45°, and change the spin orientation to "$+z$". You now have the simulation set up to take atoms with $S_z = +\hbar/2$ and measure the spin of these particles along a 45° angle.

   (c) To get a feel for how this works, try firing a few atoms. You'll notice a counter at the bottom of the page keeping track of how many atoms have been measured with spin up or spin down as measured along a 45° angle.

   (d) Turn on "AutoFire" and keep it running until the percentages are no longer changing. (Increase the speed of the auto fire so that you don't have to wait forever.) Based on the results of the simulation, what is the probability of measuring $S_{45^\circ} = +\hbar/2$?

   (e) Now, using Eq. (8.18), calculate the probability of measuring a particle to have $|\nearrow\rangle$ if it started in the state $|\uparrow\rangle$. Do your results agree with those of the simulation?

8. Einstein died in 1955, before Bell had derived his theorem and well before Aspect did the experiments that ruled out hidden variable theories. It is often stated that refusing to accept quantum mechanics is one of Einstein's mistakes. What do you think of that statement?

9. Suppose we modified Bell's experiment and aligned the positron detector to be at a 60° angle with respect to the z-axis.

(a) In quantum mechanics, if the electron is measured to be spin down ($S_z = -\hbar/2$), what is the probability that the positron will be found to have $S_{60^\circ} = +\hbar/2$?

(b) Now we turn to the hidden variable theory of Section 8.6. In this case, if the electron is measured to be spin down what is the maximum probability that the positron will be found to have $S_{60^\circ} = +\hbar/2$?

(c) Note the difference between the predictions of quantum mechanics versus the hidden variable theory in this case.

10. Explain why hidden variable theories do not require "spooky action at a distance."

11. Show that the state

$$|\psi\rangle = \frac{1}{2}\,|{\uparrow\uparrow}\rangle - \frac{1}{2}\,|{\uparrow\downarrow}\rangle + \frac{1}{2}\,|{\downarrow\uparrow}\rangle - \frac{1}{2}\,|{\downarrow\downarrow}\rangle$$

is separable and can be written as $|\psi\rangle = |\phi_{\text{electron}}\rangle\,|\phi_{\text{positron}}\rangle$. Find $|\phi_{\text{electron}}\rangle$ and $|\phi_{\text{positron}}\rangle$.

12. An electron-positron pair is in the state

$$|\psi\rangle = 0.5\,|{\uparrow\uparrow}\rangle + 0.1\,|{\uparrow\downarrow}\rangle + 0.7\,|{\downarrow\uparrow}\rangle + 0.5\,|{\downarrow\downarrow}\rangle.$$

(a) Convert this state to the form $|\psi\rangle = c_+\,|{\uparrow}\rangle\,|\phi_1\rangle + c_-\,|{\downarrow}\rangle\,|\phi_2\rangle$. Determine the coefficients c_+ and c_- and the normalized positron states $|\phi_1\rangle$ and $|\phi_2\rangle$.

(b) Based on your answer to part (a), is $|\psi\rangle$ entangled or not? Describe how you can tell.

13. An electron-positron pair is in the state

$$|\psi\rangle = \frac{1}{\sqrt{10}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{5}}|\uparrow\downarrow\rangle + \frac{1}{\sqrt{5}}|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle$$

which is the same state as in Problem 3.

(a) Determine the probability of a positron S_z measurement finding the value $+\hbar/2$.

(b) Starting from state $|\psi\rangle$, an electron S_z measurement has found the value $+\hbar/2$. Using your results from Problem 3, calculate the probability of the positron being spin up.

(c) Starting from state $|\psi\rangle$, an electron S_z measurement has found the value $-\hbar/2$. Using your results from Problem 3, calculate the probability of the positron being spin up.

14. For each of the following states, determine whether they are separable or entangled.

(a) $|\uparrow\uparrow\rangle$

(b) $|\downarrow\downarrow\rangle$

(c) $\frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle$

15. Suppose we modified Bell's experiment and aligned the positron detector to be in the $+x$ direction. That is, we will measure the S_x value for the positron.

(a) In quantum mechanics, if the electron is measured to be spin down ($S_z = -\hbar/2$), what are the probabilities that the positron will be found to have $S_x = \pm\hbar/2$?

(b) Now we turn to the hidden variable theory of Section 8.6. In this case, if the electron is measured to be spin down what are the probabilities that the positron will be found to have $S_x = \pm\hbar/2$?

(c) Will this modification of Bell's experiment allow us to distinguish between quantum mechanics and our hidden variable theory?

Chapter 9

'Elementary' Particles and Conservation Laws

9.1 Introduction

One of the deepest and most persistent questions in physics is "What is the universe made of?" In this chapter we begin to address this question. The answer seems to be *particles*, and the harder we bang protons or electrons together, the more new types of particles appear. In this chapter, we discuss our current understanding of the fundamental building blocks of nature. Specifically, we discuss categories for particle types, some of their properties, and the way particles behave in various reactions. And most importantly, you'll see how conservation laws begin to make sense of the jumble of "elementary" particles. In the next two chapters, we will discuss how these particles interact with each.

A word about the term *elementary particle*: ideally, an elementary particle has a set of fixed properties, like mass, spin, charge, etc., and cannot be further subdivided into smaller constituents. Atoms were once considered elementary until they were found to be made of protons, neutrons, and electrons. Only a few of the particles discussed in this chapter are truly elementary. Thus the use of quotes.

9.2 Particle Types

It appears that on the fundamental level, all matter consists of particles. But what are these particles? How can they be distinguished and identified? Are there a few basic types from which all matter is constructed? These are some of the questions to be addressed in this section.

Now to the question of identifying particles. Basically, a particle is characterized by its mass and charge, but there are many other properties that contribute to identification. In the relativistic momentum and energy lab from Physics 211, you calculated the mass and charge of an invisible particle by invoking the conservation laws of energy and momentum. You studied the reaction

$$\pi^- + p \rightarrow \pi^- + \pi^+ + X \tag{9.1}$$

and determined that particle X had a mass about as big as a proton, but was charge neutral. This was sufficient information to identify X as a neutron.

In the 1930's, the only particles known were protons, neutrons, electrons, and photons. As better detectors became available, new particles were identified from reactions initiated by cosmic rays and as products of radioactive decay. For instance, when cosmic rays interact with gas in the upper atmosphere *muons* are created. These are negatively-charged particles that decay quickly into electrons, releasing large amounts of energy by the reaction $\mu^- \rightarrow e^- + \text{energy}$. Muons are not a part of normal matter and their existence was a puzzle for physicists.

Another milestone in particle physics was the discovery of the positron, also known as an antielectron. In 1932, Carl Anderson succeeded in identifying a positive particle in cosmic rays with the same mass as an electron. Using a cloud chamber in a magnetic field, he photographed an electron-like track being deflected in the opposite direction that one would expect from an electron. This was the first direct evidence of an *antiparticle* and served to confirm Dirac's theoretical prediction that all particles have antiparticles. Since then antiparticles have been found for essentially all known particles. Antiparticles have the same mass and spin as their corresponding particles, but take the opposite sign for additive properties like charge.

Useful as these studies were, the experimental physicists' dependence on naturally occurring energetic particles was very restrictive. In the 1940's and 50's, particle accelerators were developed that enabled physicists to extend vastly the range of energies and particles available for study. Particles like pions (π) and neutrinos (ν) were found in reactions such as

$$p + n \rightarrow n + n + \pi^+, \tag{9.2}$$

and

$$n \rightarrow p + e^- + \nu_e. \tag{9.3}$$

Most of the particle physics of the 1950's and 1960's consisted of banging particles together with more and more energy and seeing what came out. With higher energy accelerators, more and more kinetic energy of the incident particle could be converted to rest energy of new particles. An explosion of new particles, with names like lambda (Λ), sigma (Σ), kaon (K), tau (τ) and omega-minus (Ω^-), appeared on the scene (see Tables 9.1 through 9.3). From the three or four particles known in the first third of the 20^{th} century, the list of "elementary" particles had grown to several hundred. Fermi is said to have remarked that if he had known that there were so many particles whose properties he was expected to memorize, he would have taken up botany![1]

A list of particles known today would extend to many hundreds of entries. The problem of devising separate symbols to name the particles long ago exhausted the resources of the Greek and Latin alphabets and numbering particles with their masses is now commonplace. A database maintained by the *Particle Data Group* (see `http://pdg.lbl.gov/`) contains entries like $\Delta(1910)$ or $f'(1523)$, with a listing of the particle's properties (mass, charge, spin, lifetime, etc.). But to make any sense of this huge list of particles and properties, we must look at the classification schemes available.

9.3 Classification of Particles

Historically, particles were classified by their masses. Two particles with exactly the same mass are almost surely identical, unless some other obvious difference (like different charges) turns up. Even then, physicists give the particles the same symbol (like e^- for electrons, e^+ for positrons) and look for some underlying symmetry that causes them to have the same mass.

The first rough classification assigned the labels *leptons*, *mesons*, and *baryons* to particles, names derived from the Greek words for light, medium, and heavy. Thus low mass particles like electrons were called leptons, middle mass particles like pions were called mesons, and heavy particles (protons and neutrons) were called baryons.

As more particles entered the scene, a scheme based just on mass became inadequate. For instance, a muon has a mass close to that of a pion, but behaves much more like an electron. The present classification of particles is now based on two factors: the type of force the particle "feels,"[2] and the spin of the particle. Apparently all particles feel the weak interaction, but only some can feel the strong interaction. The constituents of matter that feel the strong interaction are called *hadrons*; those that do not are *leptons*. (Messenger particles,[3] like photons and W's, are neither; they are not constituents of matter but force carriers.) As for a classification based on spin, those particles with integral spin are called *bosons*, those with half-integral spin are called *fermions*, as we have seen. Let's now look at the present definitions of leptons, mesons, and baryons.

Leptons. Leptons are those particles that do not participate in the strong interaction. All leptons have spin-$\frac{1}{2}$ and are thus fermions. Their masses range from 0 to $1.78\,\mathrm{GeV}/c^2$. Leptons appear to have no internal

---

[1] Halliday and Resnick, Fundamentals of Physics Extended, Third Edition, John Wiley and Sons (1988), p. 1126.

[2] We will be discussing the fundamental forces/interactions in the next chapter. For the moment, though, you should be aware that there are four fundamental forces: the *strong* force, the *electromagnetic* force, the *weak* force and the *gravitational* force.

[3] We will be discussing messenger particles in the next chapter as well.

Table 9.1: The leptons

Symbol/Name		Mass (MeV/c^2)	Spin	Additive Q	L_e	L_μ	L_τ	Anti-particle
e^-	electron	0.511	1/2	-1	1	0	0	e^+
ν_e	electron-neutrino	~ 0	1/2	0	1	0	0	$\overline{\nu}_e$
μ^-	muon	106	1/2	-1	0	1	0	μ^+
ν_μ	mu-neutrino	~ 0	1/2	0	0	1	0	$\overline{\nu}_\mu$
τ^-	tau	1777	1/2	-1	0	0	1	τ^+
ν_τ	tau-neutrino	~ 0	1/2	0	0	0	1	$\overline{\nu}_\tau$

Notes:

- All leptons have $B = 0$ and $S = 0$.
- For antiparticles, the signs of the additive properties are reversed.

structure (down to 10^{-18} m at least) and thus appear to be true elementary particles. There are only a handful of different kinds and they come in just three groups, or families, as seen in Table 9.1. Some examples are electrons, neutrinos, and muons.

Mesons. Mesons are those particles that participate in the strong interaction and have integer spin. That is, those hadrons that are also bosons are called mesons. Their masses range from 0.135 GeV/c^2 up to 11 GeV/c^2 or more, and more are still being discovered. Only a few have lifetimes longer than about 10^{-20} s. Unlike leptons or baryons the number of mesons is not conserved — mesons can be created or destroyed in reactions. Pions and kaons are two examples of meson types. Table 9.2 lists several others.

Baryons. Baryons are those particles that participate in the strong interaction and have half-integer spin. That is, hadrons that are also fermions are called baryons. As a class, they are the heaviest and most numerous type of particle with hundreds of members and masses ranging from 0.938 GeV/c^2 up to 2.6 GeV/c^2. Both mesons and baryons seem to have a size and internal structure on the scale of 10^{-15}–10^{-16} m. Baryons are "proton-like," in that all baryons eventually decay into protons, the lightest baryon. Some other baryons are the neutron, the lambda (Λ) and the omega minus (Ω^-). A few baryons are listed in Table 9.3.

9.4 Conservation Laws

Let's now try to analyze a simple reaction like you might see in a bubble chamber, in which part of the incoming energy is used to create a new particle, say a pion.

$$p + p \rightarrow p + p + \pi \tag{9.4}$$

Table 9.2: Selected mesons

Symbol/Name		Mass (MeV/c^2)	Spin	Additive Q	B	S	Antiparticle
π^0	pion	135	0	0	0	0	π^0
π^+	pion	140	0	+1	0	0	π^-
K^+	kaon	494	0	+1	0	+1	K^-
K^0	kaon	498	0	0	0	+1	$\overline{K}^0$
η	eta	548	0	0	0	0	η
η'	eta-prime	958	0	0	0	0	η'
ρ^+	rho	775	1	+1	0	0	ρ^-
ρ^0	rho	775	1	0	0	0	ρ^0
ω	omega	783	1	0	0	0	ω

Notes:

- All mesons have $L_e = L_\mu = L_\tau = 0$.
- For antiparticles, the signs of the additive properties are reversed.
- Some mesons are their own antiparticles.

To analyze this reaction, we use conservation laws. Three conservation laws of particle physics have analogs in classical physics: conservation of charge, conservation of energy, and conservation of angular momentum. First use the law of conservation of charge, which states that in any reaction the total charge before and after the reaction must be equal. Protons carry a single unit of positive charge, but pions come in three varieties: π^+, π^0, and π^-. We can thus use conservation of charge to determine which variety of pion appears in the reaction Eq. (9.4). The answer is the neutral pion, π^0.

Next, let's see how to apply the conservation of energy. Here is a caution about applying energy conservation to reactions in which *two or more* particles are present initially. Unless you are told about the incident particles' kinetic energies, energy conservation alone cannot rule out any reaction. But in a decay, when only a *single* particle is present before the reaction, energy conservation requires that the decay products be less massive than the original particle. In a decay, a single unstable particle disappears to be replaced by two or more less massive particles. (Can you see why at least two particles are needed?)

Here are some typical decays. A positive pion can decay into an antimuon (μ^+) and a mu-neutrino (ν_μ) with a mean life of 2.6×10^{-8} s:

$$\pi^+ \to \mu^+ + \nu_\mu \tag{9.5}$$

Table 9.3: Selected baryons

Symbol/Name		Mass (MeV/c^2)	Spin	Additive Q	B	S
p	proton	938.3	1/2	+1	1	0
n	neutron	939.6	1/2	0	1	0
Λ	lambda	1116	1/2	0	1	−1
Σ^+	sigma	1189	1/2	+1	1	−1
Σ^0	sigma	1192	1/2	0	1	−1
Σ^-	sigma	1197	1/2	−1	1	−1
Δ^{++}	delta	1232	3/2	+2	1	0
Δ^+	delta	1232	3/2	+1	1	0
Δ^0	delta	1232	3/2	0	1	0
Δ^-	delta	1232	3/2	−1	1	0
Ξ^0	cascade	1315	1/2	0	1	−2
Ξ^-	cascade	1322	1/2	−1	1	−2
Σ^{*+}	sigma-star	1383	3/2	+1	1	−1
Σ^{*0}	sigma-star	1384	3/2	0	1	−1
Σ^{*-}	sigma-star	1387	3/2	−1	1	−1
Ξ^{*0}	cascade-star	1532	3/2	0	1	−2
Ξ^{*-}	cascade-star	1535	3/2	−1	1	−2
Ω^-	omega-minus	1672	3/2	−1	1	−3

Notes:

- All baryons have $L_e = L_\mu = L_\tau = 0$.
- An antiparticle is indicated with a bar, e.g., $\overline{p}$ is an antiproton.
- For antiparticles, the signs of the additive properties are reversed.

The muon is also unstable and decays on the average after 2.2×10^{-6} s:

$$\mu^+ \rightarrow e^+ + \nu_e + \overline{\nu}_\mu \tag{9.6}$$

Finally let's see how to apply angular momentum conservation. In particle physics, angular momentum is quite complicated, since it is a quantized vector quantity that includes contributions from both spin and orbital angular momentum. A complete description of this conservation law would thus require techniques beyond the scope of this course. However, we can derive one simple rule, easy to apply, based on the fact that an even number of half-integral spins always combine to give integral spin.

The Fermion Rule for angular momentum conservation: A reaction can only occur if the number of fermions before the reaction and the number after are either both odd or both even.

These three conservation laws — charge, energy, and angular momentum — are absolute: they apply in all known reactions.

In addition to the absolute conservation laws of charge, energy, and angular momentum, there are two other well-tested conservation laws, those for lepton number and baryon number. Although some modern field theories predict their violation in reactions like proton decay, these violations have never been conclusively observed. So until we discuss *grand unified theories* in Chapter 11, we will assume that lepton and baryon numbers are conserved.

So what is lepton number? Actually it's quite simple. Each lepton is assigned a lepton number of $L = +1$. The antiparticles of leptons have $L = -1$. All other particles have $L = 0$. Now consider a reaction like pion decay, Eq. (9.5). The pion is not a lepton, so it has $L = 0$. The μ^+ is the antiparticle of the μ^-, a lepton, so μ^+ has $L = -1$. Finally, a neutrino is a lepton, with $L = 1$. Lepton number is an additive quantity, so simple addition shows that both sides total $L = 0$, and the decay is allowed.

Actually, conservation of lepton number is slightly more complicated than this simple example implies. Consider, for example, the following muon decay

$$\mu^+ \rightarrow e^+ + \nu_? + \overline{\nu}_?, \tag{9.7}$$

and a possible alternative decay

$$\mu^+ \rightarrow e^+ + \gamma. \tag{9.8}$$

The second decay seems simpler and also appears to conserve lepton number, but in fact, a decay like Eq. (9.8) has never been observed. A situation like this is a puzzle for physicists. Why doesn't Eq. (9.8) ever occur? There must be something new going on that prevents it. This "something new" is the existence of a previously undiscovered conservation law!

The new law is this: Each of the three types of leptons is conserved separately. Thus, there are electron neutrinos (ν_e), muon neutrinos (ν_μ), and tau neutrinos (ν_τ). The electron and its neutrino each carry $L_e = +1$, $L_\mu = L_\tau = 0$, and similarly for the other leptons. Thus, the muon decay of Eq. (9.6) is actually

$$\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu \tag{9.9}$$

and the candidate decay Eq. (9.8) is ruled out because neither L_e nor L_μ is conserved. So when we say "conserve lepton number," it really means "conserve all three types of lepton number: L_e, L_μ, and L_τ." **REMEMBER: There are THREE separate lepton numbers, and each needs to be conserved separately.** E.g., if $L_e = 1$ and $L_\mu = L_\tau = 0$ before a reaction, and $L_e = 0, L_\mu = 1$, and $L_\tau = 0$ after the reaction, then this violates lepton number conservation since L_e isn't conserved and L_μ isn't conserved.

Similar observations lead to the formulation of the law of conservation of baryon number. Consider the two candidate reactions below:

$$\pi^0 \to \gamma + \gamma \tag{9.10}$$

and

$$n + n \to \gamma + \gamma \tag{9.11}$$

Both are allowed by all the conservation laws described so far, but the first occurs all the time, while the second never occurs. Why not? A new conservation law must be operating: conservation of baryon number. Neutrons are baryons, so they each have $B = 1$. Photons and pions are not baryons so $B = 0$ for them. Any antiparticles of baryons have $B = -1$. Finally, since baryon number is additive, we have that the decay in Eq. (9.10) is allowed because $0 = 0$, while the reaction in Eq. (9.11) is forbidden because $2 \neq 0$.

These sorts of *ad hoc* conservation laws seem to be begging the question. Later on we'll encounter a deeper reason for these laws, especially the law of conservation of baryon number.

9.5 Strangeness

We now come to the first conservation law that is partially violated: conservation of strangeness. Strangeness is a property carried only by hadrons, and it is conserved during strong and electromagnetic interactions, but not necessarily conserved in weak interactions. Its name arises from the strange behavior observed in the production and decays of certain hadrons. Consider the following sequence, a result of bombarding stationary protons with high energy pions:

$$\pi^- + p \to \Lambda + K^0, \tag{9.12}$$

followed by

$$\Lambda \to p + \pi^- \qquad \text{lifetime} \sim 3 \times 10^{-10}\,\text{s} \tag{9.13}$$

and

$$K^0 \to \pi^+ + \pi^- \qquad \text{lifetime} \sim 10^{-9}\,\text{s}. \tag{9.14}$$

Particles like Λ and K always seem to be produced in pairs. One never sees either of the following candidate reactions.

$$\begin{aligned} \pi^- + p &\to n + K^0 \\ \pi^- + p &\to \Lambda + \pi^0. \end{aligned} \tag{9.15}$$

Let's analyze these reactions. Since the Λ eventually decays to a proton, it must be a baryon, while K decays only to pions, marking the kaon as a meson. Then the reactions in both Eq. (9.12) and Eq. (9.15) conserve charge, spin, and baryon number, but Eq. (9.15) never occurs. Why not? The answer once again is that a new conservation law is being violated. Lambdas and kaons must carry a new property (in equal but opposite amounts) that pions and protons don't have. This new property, which particle physicists of the 1960's named *strangeness*, comes in integer amounts. The kaon, by convention, has $S = +1$ and the lambda has $S = -1$. Strangeness then balances in Eq. (9.12) but not in Eq. (9.15). This, then, would "explain" why the reactions in Eq. (9.15) are never observed.

But what about the decays of Eq. (9.13) and Eq. (9.14)? Don't they violate strangeness conservation? The answer is yes, they do. But these decays proceed by the weak interaction. The clue that these are weak decays is the particles' relatively long lifetimes, a topic we discuss further in Chapter 11. Suffice it to say that Λ and K^0 are produced together by the strong interaction, conserving strangeness, but they decay separately by the weak interaction, which does not conserve strangeness.

9.6 The Eightfold Way and Quarks

In the 1960's, newer and bigger accelerators came on line, and the proliferation of newly created particles continued. There are now literally hundreds of so-called elementary particles. A classification scheme was needed, and in 1962 Murray Gell-Mann and Yuval Ne'eman proposed the Eightfold Way. Shortly thereafter, Gell-Mann suggested that the various hadrons were actually composed of yet smaller entities called quarks.

Gell-Mann and Ne'eman studied the relationships among various groups of particles. They were looking especially for ways to organize the hadrons into groups with common dynamical properties. For instance, particles that differ only in charge (and slightly in mass) are already given the same symbol (like Σ^+, Σ^0, and Σ^-). Experimental evidence indicates that these particles behave identically during strong interactions. There is also good evidence from nuclear physics that the strong interaction does not depend on a particle's charge. Another way to describe this independence is to say that the strong interaction possesses charge symmetry.

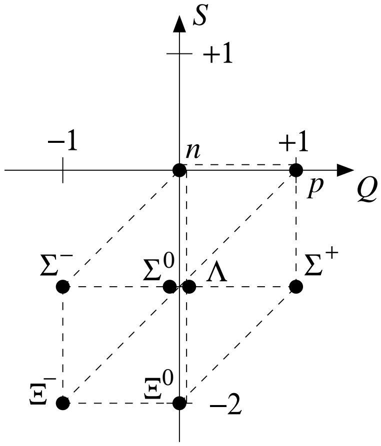


Figure 9.1: Strangeness vs. charge plot for the eight spin 1/2 baryons.

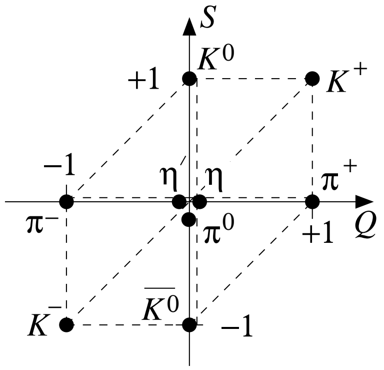


Figure 9.2: The nine spin zero mesons on a strangeness vs. charge plot.

Another common dynamical property among hadrons seems to be strangeness. Particles differing only in strangeness (and slightly in mass) are observed to behave identically during strong interactions. Thus, the strong interaction also possesses strangeness symmetry.

Let's see how Gell-Mann and Ne'eman used these symmetry ideas to group the hadrons. The classification scheme they devised, called the Eightfold Way, grouped together hadrons having the same spin and baryon number but different charge and strangeness. Because of charge and strangeness symmetry, the particles within one group should be interchangeable with respect to their behavior in strong interactions and have masses in a fairly small range.

As an example, let's look at the eight known spin 1/2 baryons. The upper part of Table 9.3 lists their properties. They all have the same spin and baryon number, and their masses are in a fairly small range (about 0.940 to $1.340\,\mathrm{GeV}/c^2$). They do differ in charge and strangeness however. When one plots the particles on a strangeness versus charge plot, a distinctive hexagonal pattern emerges. See Fig. 9.1.

Gell-Mann and Ne'eman noticed that this hexagonal pattern is the same as that which arises in the mathematical *group* of 3×3 matrices called SU(3). An important property of these matrices is that they can all be expressed in terms of eight special basis matrices. Although a detailed explanation of the theory of groups is far beyond the scope of this text, some development of the correspondence between particle plots and matrices is in order.

If we look at the strangeness versus charge plot, we find that there are certain motions that go from one particle to another. One can move along any of the three directions parallel to the sides of the hexagon. The two horizontal motions (left or right) correspond to the symmetry that increases or decreases the charge by one unit. The two vertical motions (up or down) correspond to increasing or decreasing the strangeness by one unit. Finally, the diagonal motions will change both charge and strangeness by one unit. These six operations, plus one for measuring a particle's charge and another for measuring strangeness, correspond to the eight basis matrices for the group of 3×3 matrices, and lead to the name Eightfold Way. The Eightfold Way is also a term from Eastern philosophy and refers to the steps along the Buddhist path to enlightenment. But don't read too much into the name: Gell-Mann was just having fun with it.

Let's see how the Eightfold Way gives some structure to a different set of particles, the nine known spin zero mesons. Again, locating the particles on a strangeness versus charge plot shows the hexagonal pattern. The operations that change or measure charge and strangeness are similar to those for the baryons.

It turns out that all the known particles fit into similar patterns. In fact, Gell-Mann was so convinced that his patterns were correct that, when he found a hole in the spin 3/2 baryons' pattern, he predicted the existence of a new particle and described its properties. Within weeks of his 1962 prediction, the Ω^- particle, with the correct properties, was discovered in bubble chamber photographs at Brookhaven National Laboratory, providing strong experimental confirmation of Gell-Mann's theory.

9.7 Quarks

Nice as these patterns of the Eightfold Way are, many questions remain. For instance, why are there two particles at the center of the baryon hexagon of Fig. 9.1, and three in the center of the meson pattern in Fig. 9.2? Why don't other patterns occur?

The fact that the elementary particles fall into neat patterns is reminiscent of the construction of the periodic table of elements. There it was found that when elements were arranged in (approximate) order of increasing mass number, certain groups of elements (the columns of the table) emerged with similar chemical properties. Once the periodic table was well established, the jump was quickly made to understanding atomic structure, with electrons in shells, subshells, and orbitals.

This idea that regular patterns of particle properties could indicate a regular underlying structure occurred to Gell-Mann, and in 1964 he proposed the *quark model* of elementary particle structure. Gell-Mann's quark model states that all hadrons are composed of a small number of constituent particles called quarks. The quarks of Gell-Mann's model come in three different varieties (or flavors) called up, down, and strange. The quarks' properties are summarized in Table 9.4. The vast variety of hadrons comes about as different combinations and configurations of the quarks, just as the hundreds

Table 9.4: Quark properties

Flavor	Name	Spin	Q	B	S
u	up	1/2	+2/3	1/3	0
d	down	1/2	−1/3	1/3	0
s	strange	1/2	−1/3	1/3	−1

Note: the antiquarks ($\overline{u}$, $\overline{d}$, $\overline{s}$) have the same spin but the opposite Q, B, and S.

of different isotopes of atoms arise from different combinations of protons, neutrons, and electrons.

The mass of a hadron is not simply the sum of the masses of its constituent quarks – the energies of the bound quarks can affect the mass of the hadron.[4] Most of our discussion will focus on quarks combined together in ground state configurations. But it is possible for the same set of quarks to be configured into higher energy states (i.e., higher mass particles), which usually decay quickly. (Since Gell-Mann's original model, three more flavors of quarks have been identified, but we will not study them in detail.)

Let's see how different combinations of quarks can be assembled to make various particles. Consider a neutron, a neutral, non-strange baryon. Whenever we combine quarks to make a particle, the particle's additive properties (Q, B, and S) are simply the sum of the constituent quarks' properties. Since quarks have $B = 1/3$, we need at least three quarks to make a baryon. let's try to make a neutron from just three. If we use a strange quark, the resulting particle would carry strangeness. So to make a neutron, we should use only ups and downs. But how many of each? We can find out by making the charge come out to zero: the result is one up and two downs. Thus, in symbols, $n = (udd)$.

To make a meson is even simpler. Since the baryon number must come out zero, a quark plus an antiquark will do it. Let's make a positive pion, a non-strange meson. Since $S = 0$, we could try $\pi^+ = (s\overline{s})$, but then the charge would be $-\frac{1}{3} + \frac{1}{3}$, which isn't right. Thus pions are constructed using just *ups* and *downs*.

We can summarize our findings about constructing hadrons out of quarks with the following:

---

[4] Don't forget the equivalence of energy and mass from relativity.

baryons contain three quarks (qqq)

antibaryons contain three antiquarks ($\overline{qqq}$)

mesons contain a quark-antiquark ($q\overline{q}$)

The quark model is especially convincing because it explains so well the Eightfold Way patterns of particles. Consider again the spin zero mesons. Figure 9.2 showed a hexagonal pattern with three particles in the center. When we make a similar plot of the nine possible quark-antiquark pairs, we get the pattern shown in Fig. 9.3. There are nine combinations in exactly the same places on the plot as in Fig. 9.2. A similar situation occurs when we make three-quark combinations for baryons. In fact, every possible combination of up, down, and strange quarks has been observed as a particle, and every known hadron can be built of quarks.

Once one accepts the idea of quarks, the strong interaction conservation laws become almost automatic. Consider the reaction

$$\pi^- + p \to \Lambda + K^0 \tag{9.16}$$

By now you could easily check that angular momentum, charge, baryon number, and strangeness are all conserved. But write the reaction in terms of quarks:

$$(d + \overline{u}) + (u + u + d) \to (d + u + s) + (\overline{s} + d) \tag{9.17}$$

Then simply note that quark conservation *implies* conservation of all the additive physical quantities, since each quark retains those properties regardless of configuration. By quark conservation, we mean that a quark can be created or destroyed only if its antiquark is created or destroyed simultaneously, as occurs with a $u\overline{u}$ pair and the $s\overline{s}$ pair in Eq. (9.17). You'll

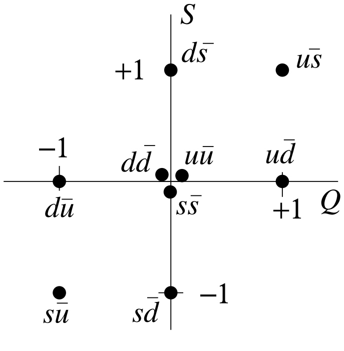


Figure 9.3: The nine quark-antiquark combinations on a strangeness vs. charge plot. Note the striking similarity to Fig. 9.2.

see this quark description of reactions more vividly when you learn about reaction diagrams in Chapter 11.

Example 1. Quark content of the Δ^{++}.

Construct from quarks a baryon with charge +2, the Δ^{++}.

Solution: Since a baryon contains three quarks, and only an up quark has a charge of +2/3, it would require three such quarks to obtain a charge of +2. Thus, the Δ^{++} must be made of (uuu).

Problems

1. Baryons and leptons are both fermions. In what ways are they different?

2. Mesons and baryons are each sensitive to the strong force. In what way are they different?

3. Test the conservation laws for energy, charge, baryon number, lepton number, and strangeness for each of the following proposed reactions. Which ones are violated?

(a) $\mu^- \to \pi^- + \nu_\mu$
(b) $\pi^- + p \to \Lambda + \overline{\nu}_e$

4. Classify each of the following particles as bosons, fermions, hadrons, leptons, baryons, mesons and/or messengers. Use as many of these terms as apply.

(a) photon (b) Λ (c) π^+ (d) ν_e

5. A ρ^0 particle initially at rest is observed to decay by $\rho^0 \to \pi^+ + \pi^-$. The pions have a mass of $140\,\mathrm{MeV}/c^2$. The magnitude of the momentum of each of the oppositely directed pions is measured to be $361.5\,\mathrm{MeV}/c$. Conserve energy and momentum to determine the rest energy of the ρ^0. Compare to the value given in Table 9.1.

6. Use the appropriate conservation laws in each of the strong reactions below to determine the additive properties of the unknown particle x. Then identify the x particle using Tables 9.1 through 9.3.

(a) $p + p \to p + \Lambda + x$
(b) $K^- + p \to \Xi^- + \pi^0 + x$

7. Check strangeness conservation to determine which of the following reactions may proceed by the strong interaction.

(a) $K^- + p \to \Lambda + \pi^0$
(b) $\Xi^- \to \Lambda + \pi^-$
(c) $K^+ \to \pi^+ + \pi^0$
(d) $\Delta^{++} \to p + \pi^+$

8. Use the data in Tables 9.1 through 9.3 to check that Eq. (9.5) and Eq. (9.6) do not violate the laws of conservation of energy, charge, lepton number, and baryon number.

9. Can an electron decay by disintegrating into two neutrinos? Why, or why not?

10. A neutron is massive enough to decay by the emission of a proton and two neutrinos. Why does it not do so?

11. Consider the reaction $\pi^- + p \to \Xi^0 + K^0 + K^0$. Assuming this reaction occurs via the strong interaction, determine the strangeness of the Ξ^0. Compare to the value in Table 9.3.

12. What are the quark constituents for each of the following particles?

(a) Λ (b) Ξ^- (c) π^+

13. Use the quark model to explain why there is no baryon with $S = -2$, $Q = +1$.

14. Determine the additive quantum numbers (B, Q, and S), and thus the identities, of the particles formed from the following combinations of quarks. Then use the tables in Chapter 9 to identify each particle. Note that a quark ingredient list does *not* always identify a particle uniquely. Sometimes several particles contain the same set of quarks.

(a) ddu (b) uus (c) $d\overline{s}$ (d) $\overline{u}\overline{u}\overline{d}$ (e) ssd (f) $u\overline{d}$

15. Consider the ten baryons with spin 3/2 listed in Table 9.3. Make a strangeness vs. charge plot for these baryons, similar to Fig. 9.1. Compare your plot with that in Fig. 9.1.

16. Consider the reaction $\Lambda + \pi^- \to \text{baryon} + \text{meson}$. What possible baryon-meson pairs can be produced? Consider only the simplest cases, in which no quarks are created or destroyed, but merely rearrange themselves.

Chapter 10

Fundamental Forces and Interactions

10.1 Introduction

If the universe is like a movie, then the elementary particles that we discussed in the previous chapter are like the cast for this movie. But what if actors in a movie didn't interact at all with each other or with anything else in the movie? Imagine that you went to see *Star Wars: The Rise of Skywalker.* The movie started and you saw Daisy Ridley, John Boyega, Adam Driver, a CGI version of Carrie Fisher and a cute, beach-ball-like android on the screen, but they didn't do anything. They didn't interact at all with each other. They didn't talk with each other. They didn't fly cool spaceships from one planet to another or engage in long fight scenes with conveniently-colored light sabers. They didn't fight against an evil Emperor who **supposedly** had died in a previous trilogy and for some ridiculous reason was back again. And they didn't inexplicably kiss each other at the end of the movie. Nothing at all happened – they just sat there. An interesting movie it would not be.[1]

That's what the universe would be like without any forces or interactions. Only worse. A lot worse. The universe would be a really boring place without any interactions. Nothing of any interest would happen. In fact, it would be **so** boring that there wouldn't even be anyone to realize just how bored they were.

Forces probably rank #1 on the list of things that we most take for granted. Particles are all very nice, but if they don't interact with anything else, then for all intents and purposes they might as well not even exist.[2] We would never be able to detect the effects of any particle that didn't experience at least *some* kind of force/interaction.

---

[1] Come to think of it, that probably would have been better than the actual movie, which was **so** bad that it was actually quite entertaining to see how bad it could be.

[2] In fact, you could have an interesting debate as to what it really *means* for something to exist, but we won't go into that here.

In this chapter, we discuss our current understanding of the fundamental forces of the universe. We describe the four fundamental forces – strong, electromagnetic, weak, and gravitational – and discuss efforts being made by physicists to describe all of these forces under the framework of a single, unified theory. We then discuss Richard Feynman's quantum field theory that describes *how* forces might actually *work*, and apply these approach to the electromagnetic and strong forces. (We will extend the approach to the weak force in the next chapter.)

10.2 The Four Fundamental Forces of Nature

Throughout the year, we have discussed a wide variety of forces: tension forces, normal forces, friction forces, air and fluid drag, spring forces, gravitational forces, electric and magnetic forces, etc. But most of these forces are just different manifestations of the same fundamental force. For instance, a normal force between two objects "in contact with each other"[3] is just electrostatic repulsion between the electrons at the surface of one object and those at the surface of the other object.

During the past century, physicists have typically talked about four "fundamental" forces: the strong force, the electromagnetic force, the weak force, and the gravitational force. You are already familiar with the electromagnetic force (which affects any particle with an electric charge) and gravitational force (which affects any particle with mass). The strong force is the force that holds protons and neutrons together in the nuclei of atoms. This is no small matter here: the electric force is extremely repulsive between protons packed together in a nucleus, so there clearly must be a force that is even stronger pulling them together. But the strong force is only active over short distances, those comparable to the sizes of atomic nuclei. For separations greater than that, the strong force is ineffective. As discussed in the previous chapter, the strong force affects all of the hadrons. Actually, since the hadrons are all made up of quarks, it can be said that the strong force is the force that is associated with quarks.

The weak force is somewhat of a "miscellaneous/etc" force; in fact, all of the elementary particles can experience the weak force. For quarks or charged particles, though, the weak force is subordinate to the strong and/or electromagnetic force(s). Only for uncharged leptons (i.e., neutrinos) is the weak force dominant. But since the weak force is **so** weak, neutrinos can pass through other objects with only minimal interactions. Right now, over 10^7 neutrinos pass through you every second. In fact, the vast majority of neutrinos that arrive at the Earth (from the Sun) pass through the entire

---

[3] We put "in contact with each other" in quotes because nothing ever really *touches* anything else on a subatomic level. The sensation or appearance of "contact" is the result of the repulsive force becoming very large.

planet without interacting with any other matter!

The weak force is also important in a lot of decay processes. For example, β-decay by which a neutron decays ($n \to p + e^- + \nu$) is a weak interaction. In fact, the weak interaction was first proposed because this decay process couldn't be explained with any of the other fundamental interactions.

10.3 Quantum Field Theory

Modern physical theory seeks to explain all four of the fundamental interactions — gravity, electromagnetism, strong, and weak — in terms of quantum fields. Here's how classical models explain how two particles exert forces on one another: each particle sets up a *field* that the other particle reacts to, and these fields obey certain equations (like Faraday's Law). This, of course, isn't really an explanation at all, because it doesn't really explain what a field is and why one particle should produce a field and why another particle should respond to this field.

In quantum field theory, the fields themselves are quantized; changes in the fields occur in discrete lumps. The lumps of energy and momentum can be considered particles, often called *messenger particles.* As an analogy, think of two people playing catch. The ball passes back and forth between the players and transmits the exchange of momentum and energy.

Let's see how this works for electric forces. The quantum field theory for electromagnetic interactions is called quantum electrodynamics or QED. In this theory an electrically charged particle — an electron, for instance — makes its presence known not by setting up an electric field, but by continually emitting and absorbing electromagnetic messenger particles, or *photons.* Because these photons are not detected as particles of light, but serve only to transmit energy and momentum between particles, they are also called *virtual photons.* In fact, at any given time, an electron is literally surrounded by a cloud of these virtual photons, appearing and disappearing.

Where does the energy come from to create all these photons? Doesn't energy have to be conserved? The answer is that short-term violations of the law of conservation of energy are permitted by the quantum uncertainty relations, as we have already seen in tunneling. The uncertainty relation for time and energy, analogous to $\Delta x\, \Delta p \geq \hbar/2$, is

$$\Delta E\, \Delta t \approx \hbar \tag{10.1}$$

In effect, Eq. (10.1) permits particles to borrow an amount of energy ΔE from the surrounding space so long as the energy is returned within a time $\Delta t \approx \hbar/\Delta E$. In this view, even empty space itself is loaded with activity, as virtual particles and messengers pop into existence and then quickly disappear![4]

---

[4] Experiments have measured these effects in "empty" space. Specifically, even a well-

Table 10.1: The messenger particles.

Symbol	Name	Mass (MeV/c^2)	Spin	Charge	Interaction
γ	photon	0	1	0	Electromagnetic
W^-	double-you	80,400	1	-1	Weak
Z^0	zee-zero	91,200	1	0	Weak
gl	gluon	0	1	0	Strong
g	graviton?	0	2	0	Gravity

Note: the antiparticle of W^- is W^+, and the antiparticle of a green-antired gluon is a red-antigreen gluon, etc.

Current theory holds that there are messenger particles like photons that act as the intermediaries for all of the fundamental interactions. For the weak interaction, they are the massive (as in, having mass) *vector bosons* W^+, W^-, and Z^0. For the strong interaction, they are the colored *gluons*, which act between quarks. The properties of the messenger particles are listed in Table 10.1. You'll learn more about these messenger particles in this and the next chapter. In addition, a particle called the *graviton* is postulated as the messenger for gravity.

10.4 Feynman Diagrams

In putting the ideas of quantum field theory together in the 1940's, Richard Feynman developed a method of diagramming interactions. These diagrams can actually be translated directly to equations using a set of Feynman rules that associates different parts of the diagram to various factors and terms in the equations. We will not try to do that, but we will use Feynman-like diagrams to sketch reactions.

Let's diagram the interaction of two electrons repelling each other via the electromagnetic interaction. We've just seen that one electron emits a virtual photon as the messenger, and the other electron receives it. If we plot time upward and space horizontally (like a space-time diagram) we find the interaction looks like Fig. 10.1.

The particles that interact are drawn as solid straight lines, indicating constant velocity before and after the interaction. The lines can change direction only at a vertex, the point where two solid lines and a squiggly photon line meet. The vertex represents the emission or absorption of the messenger particle, events which can change the properties of the interacting particles. In this case, the momentum of each electron changes at the vertex.

The weak and color interactions can also be illustrated in this way. The messenger particles for the weak interactions are the W^+, W^-, and Z^0

---

shielded vacuum is observed to have electromagnetic "noise" due to the blipping into and out of existence of these virtual photons.

bosons, which on a diagram are represented by closely spaced parallel lines. For the strong interaction between the quarks, the messengers are colored gluons, which appear as double squiggly curves. Figs. 10.2 and 10.3 show examples of weak and strong interactions.

10.5 Quark Color

At first glance, the quark model of the hadrons as outlined by Gell-Mann is very appealing in that it seems to explain the occurrence of most of the observed particles. But several problems come forward when a closer look is taken. Consider the three up quarks that make up the Δ^{++} on page 226 in Chapter 9. The Δ^{++} is the lowest mass doubly-charged baryon, so presumably the quarks are each in the ground state.[5] But quarks are fermions (spin 1/2) and so the Pauli exclusion principle should apply! How can three identical quarks all occupy the same state?

The way around this problem is to postulate that the three up quarks in Δ^{++} (or the three quarks in any baryon) are not identical, but are each a different "color." Now of course they don't actually appear colored, but for the reasons described below we call this extra property of quarks color, and a quark can be red (R) or blue (B) or green (G). The antiquarks come in anti-colors: anti-red ($\overline{R}$), anti-blue ($\overline{B}$), or anti-green ($\overline{G}$). No kidding, we really use these labels.

There is a single simple rule about quark color and here it is:

The Colorless Rule: No physically observable particle ever carries net color.

By never carrying a net color, we mean that the colors of the constituent quarks in a particle must cancel out, so that the composite particle is colorless. This can happen in three ways.

---

[5] That is, there are other, higher mass baryons with +2 charge that are also made up of three up quarks, but these correspond to excited state configurations of the quarks.

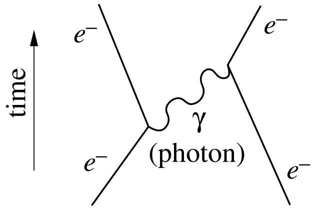


Figure 10.1: Feynman diagram of two electrons interacting electromagnetically by the exchange of a photon.

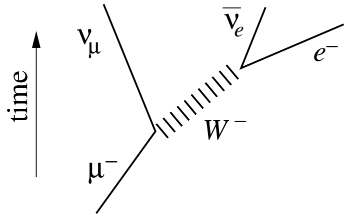


Figure 10.2: Feynman diagram showing a muon decaying by the weak interaction.

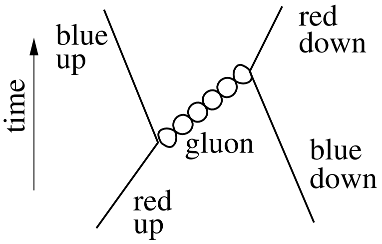


Figure 10.3: Feynman diagram showing quarks interacting strongly by exchanging a gluon.

1. A particle can contain a colored quark and an anti-quark of opposite color, i.e., $R\overline{R}$, $B\overline{B}$, or $G\overline{G}$.

2. A particle can contain three quarks, one of each color (i.e., RBG). The colors cancel out in much the same way as the three primary colors of light mix to give white (colorless) light. This is the reason we say quarks carry color rather than using some other name.

3. A particle can contain three antiquarks, one of each anti-color (i.e., $\overline{RBG}$).

The colorless rule automatically picks out those combinations of quarks that actually occur in nature. We never see a lone quark, or combinations such as qq, $q\overline{q}\overline{q}$, or $qqq\overline{q}$, because these would of necessity show net color. Furthermore, the rule automatically forbids the possibility of observing particles with fractional baryon number or fractional charge. The Rule is crafted so that we are only allowed to observe what we in fact do observe.

The colorless rule is actually a simplified way of describing a very deep symmetry in nature, *color symmetry.* Since net color can never be observed, it really can't matter whether we label the quarks in a proton as RBG or RGB or BGR or BRG or GRB or GBR. We'll see in the next section how the colorless rule is enforced by the exchange of *gluons*, the messenger particles that actually hold the quarks together and keep them confined inside hadrons.

10.6 Chromodynamics: The Theory of Colored Quarks

Let's now investigate the strong interaction between quarks. The strong interaction is actually a manifestation of the color property of quarks. Just as with electric charges, color charges repel if alike and attract if different. Now a quark, like any particle of quantum field theory, continually emits

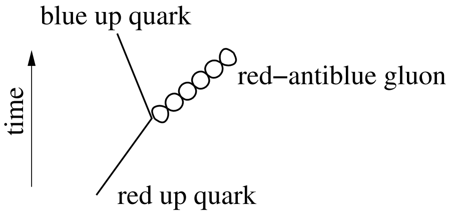


Figure 10.4: Feynman diagram showing a quark changing color as it emits a gluon.

and absorbs messenger particles and is thus surrounded by a small cloud of virtual quarks and gluons. Nearby quarks then detect the presence of the first quark by interacting with its cloud of colored gluons.

It is important to understand what happens to a quark when emitting or absorbing gluons: *as a quark emits a gluon, its color changes.* This color change can be seen in Fig. 10.4. Note that a gluon carries both a color and an anticolor. In the example, the red up quark on the left emits a red-antiblue gluon and changes into a blue up quark. You can think of this by saying that the red color leaves the quark and goes into the gluon. The antiblue color of the gluon is also withdrawn from the quark, leaving the quark blue. The gluon may then carry color to another quark, or it may return to the original quark to be re-absorbed.

An important aspect of the color force between quarks is how its interaction strength varies with distance. We can best understand this concept through an analogy with electric forces. As you recall from our study of Gauss' Law, the electric flux through a surface depends only on the *net charge* inside the enclosed volume. Similarly, the color field depends only on the *net color* inside a given volume. We must thus consider the effect of the gluon messenger particles, since they can carry the color charge away from a quark.

The color carried by a gluon leads to the effect called *camouflage.* Consider a red quark. Suppose while you are trying to test the color inside a certain region surrounding the quark, it emits a red-antigreen gluon, leaving behind a green quark. The gluon can't go far without violating the uncertainty principle and unless it finds another quark, it soon returns and is re-absorbed. But during the time it's away, the net color in a large region enclosing the fleeing gluon is still red, while in a smaller region missing the gluon, the net color of the quark looks green. The combined effect of gluons continually leaving and returning is that the closer you get to the red, the less red it appears. Thus the effective color charge decreases as you move closer to it.

Detailed calculations show that this camouflage effect makes the color charge continually decrease at smaller and smaller separations. In fact, it

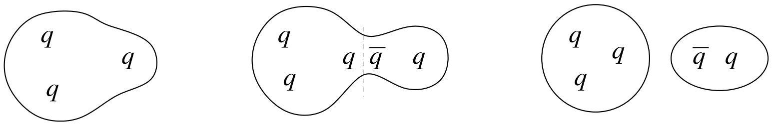


Figure 10.5: Trying to pull a quark out of a baryon fails. The energy creates a new meson instead.

turns out that on distance scales smaller than a proton radius, the effective quark color charge is so diminished that the three different colored quarks inside a proton hardly affect each other and move almost independently. The gluon force between quarks thus acts like a rubber band tied between two balls. The balls move freely until the rubber band gets stretched. Then the more they try to move apart, the harder the rubber band force pulls them back together. Continuing the rubber-band analogy, you can see that, as you try to separate quarks, the effective color charge increases rapidly and the inter quark forces become huge. In fact, the energy needed to pull apart quarks is so great that a quark-antiquark pair is created instead, as shown in Fig. 10.5. Instead of a free quark, we just create a new meson. This is why we've never observed free quarks; all quarks are permanently confined inside hadrons.

10.7 Interactions and Messenger Bosons

Before describing the weak interaction on a fundamental level, let's review how photons and gluons serve as the messengers for the electromagnetic and strong interactions between leptons and quarks. Messenger particles that transmit interactions, such as photons, gluons, and the W's and Z^0's, are also called intermediate bosons, because they mediate interactions and carry integer spin.

Consider first the most familiar lepton, an electron, emitting or absorbing

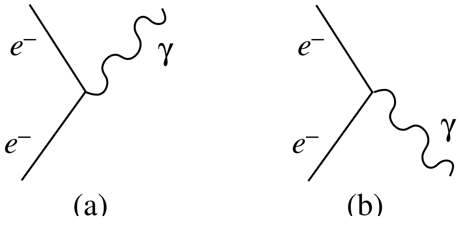


Figure 10.6: Feynman diagrams of typical electromagnetic interactions. An electron changes momentum by (a) emitting a photon ($e^- \to e^- + \gamma$), or (b) absorbing a photon ($e^- + \gamma \to e^-$).

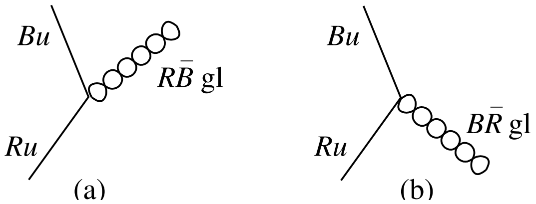


Figure 10.7: Feynman diagrams of typical quark-gluon interactions. An up quark changes color by (a) emitting a gluon ($Ru \rightarrow Bu + R\bar{B}gl$), or (b) absorbing a gluon ($Ru + B\bar{R}gl \rightarrow Bu$).

a photon. The basic interaction occurs at a vertex in a Feynman diagram. See Fig. 10.6. As the electron emits or absorbs a photon, its energy and momentum change, but it remains an electron.

Turn now to the description of a quark emitting or absorbing a gluon. See Fig. 10.7. Notice that only gluons carrying red ($R\overline{R}$, $R\overline{B}$, or $R\overline{G}$) can be emitted by red quarks, while only gluons carrying antired ($R\overline{R}$, $B\overline{R}$, or $G\overline{R}$) can be absorbed by red quarks. Similar rules apply for the gluons emitted or absorbed by blue or green quarks. As the quarks in the diagram emit or absorb a gluon, their energy, momentum, and color may all change, but the *flavor* remains the same, i.e., a down quark remains a down quark, an up quark remains an up quark, etc.

The weak interaction, mediated by the $W^{\pm}$ and Z^0 bosons, is probably the hardest one to understand. You saw in Chapter 9 that the weak interaction violates strangeness conservation, and it is known that certain other symmetries are violated as well. We know that both quarks and leptons seem to "feel" the weak interaction, but it affects quarks and leptons somewhat differently. Since the $W^{\pm}$ bosons carry electric charge, the particle emitting or absorbing a W boson must change its identity in a major way.

Consider first how W bosons interact with quarks. A W boson *changes a quark's flavor.* That is, if an up quark emits or absorbs a W, then it is no longer an up quark. In Fig. 10.8, we see a red strange quark emitting or absorbing a W and becoming a red up quark. Its energy, momentum, charge and flavor change, but the quark color remains red.

While Fig. 10.8 shows typical interactions at a quark-W vertex, we can derive several other versions based on the symmetry properties of Feynman diagrams. Look at Table 10.2. Each operation is based on the standard version. Also, any *combination* of these operations yields a valid quark-W interaction.

Finally, let's look at how W's interact with leptons. A W boson *changes a lepton's "family membership."* We saw earlier that there are three types or families of leptons: electron, muon, and tauon. Each family contains a negative particle, a positive antiparticle, and a neutrino and antineutrino. The basic interaction is that a lepton emits a W^+ or W^- and changes into a different lepton of the same family. This process of changing to a different lepton of the same family is what we mean by the phrase "change of family membership." As Fig. 10.9 illustrates, the electron emits a W^- or absorbs a W^+ and changes its charge and family membership, becoming an electron neutrino. But, it maintains its lepton number, $L_e = +1$.

As with the quark interactions, we can derive several versions of the lepton-W interaction. These are listed in Table 10.3. Each operation is based on the standard version. Again, any combination of these operations yields a valid lepton-W interaction.

An important thing to note here is the equivalence in these processes and in the diagrams between a particle "going forward in time" and an

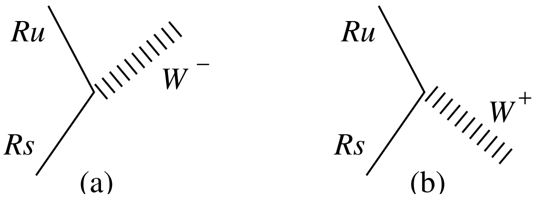


Figure 10.8: Feynman diagrams of typical quark-W boson interactions. A strange quark changes flavor by (a) emitting a W^- ($Rs \to Ru + W^-$), or (b) absorbing a W^+ ($Rs + W^+ \to Ru$).

Table 10.2: Weak interactions of quarks.

standard version	$s \to u + W^-$
reverse arrow	$u + W^- \to s$
particle out = anti-particle in	$\overline{u} + s \to W^-$
change to antiparticles	$\overline{s} \to \overline{u} + W^+$
replace s with d	$d \to u + W^-$

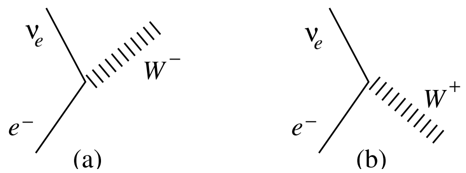


Figure 10.9: Feynman diagrams of typical lepton-W boson interactions. An electron changes its family membership by (a) emitting a W^- ($e^- \to \nu_e + W^-$), or (b) absorbing a W^+ ($e^- + W^+ \to \nu_e$).

Table 10.3: Weak interactions of leptons.

standard version	$e^- \to \nu_e + W^-$
reverse arrow	$\nu_e + W^- \to e^-$
particle in = antiparticle out	$\overline{\nu}_e + e^- \to W^-$
change to antiparticles	$e^+ \to \overline{\nu}_e + W^+$
replace e with μ or τ	$\mu^- \to \nu_\mu + W^-$

antiparticle "going backward in time."[6] In Fig. 10.9, for instance, an electron is changed into a ν_e neutrino either by emitting a W^- or by absorbing a W^+. This can result in some ambiguity: a process in which particle A emits a W^-, which is then absorbed by particle B, could just as easily be re-written as a process in which particle B emits a W^+, which is then absorbed by particle A.

---

[6] As an example of this seemingly-crazy idea, try drawing a Feynman diagram for the annihilation of an electron and positron, resulting in the production of two photons. Mathematically, you could describe the same process as an electron moving along, emitting two photons, and then going backwards in time. (Or, alternately, an electron emits a photon, gets deflected, then emits another photon, and then goes backwards in time.) Note that this does not mean that quantum theory predicts the possibility of macroscopic objects traveling backwards in time — it does not.

Problems

1. Think about what you are doing (or not doing) right at this minute. (a) Identify (and write down) as many forces as you can that are acting on you right now and/or are affecting your environment. (b) Describe what your life would be like if there were no forces or interactions at all.

2. (a) What is the minimum energy fluctuation ΔE required to create a virtual electron-positron pair? (b) Roughly how long could such a virtual pair survive?

3. The Z-boson (one of the mediators of the weak force) has a mass of $91.2\,\mathrm{GeV}/c^2$.
    (a) What is the minimum energy fluctuation required to create a virtual Z-boson?
    (b) Roughly how long could such a virtual Z survive?
    (c) What is the farthest that this particle could travel during this lifetime?
    (d) How does this result compare with the known range of the weak force ($\sim 10^{-3}\,\mathrm{fm}$)?

4. Several years ago, a group of physicists claimed to have discovered a fifth force with a range of approximately 100 m. If this force had turned out to be valid (few people believe this to be the case), what would have been the approximate (order of magnitude) mass of its messenger particle? Express your answer in eV/c^2.

5. Draw a Feynman diagram corresponding to the annihilation of an electron-positron pair. An electron moves to the right and hits a positron moving to the left. They annihilate each other, resulting in the formation of two photons, one moving left and the other moving right. *Hint:* You will need two vertices.

6. A Feynman diagram can be played with like a Gumby doll. You can bend all the branches up and down, as long as you follow one simple rule: every time you bend an upward-going branch downward (or a downward-going branch upward), you replace the particle with its antiparticle. For example, in Fig. 10.2, if the ν_μ branch is bent downward, you can get the interaction $\overline{\nu}_\mu + \mu^- \to \overline{\nu}_e + e^-$.
    (a) Verify that this new reaction satisfies all the appropriate conservation laws.
    (b) Use the Gumby approach to come up with several other possibilities for interactions, starting with Fig. 10.2.

7. Show with some examples that the colorless rule yields particles with integer charge.

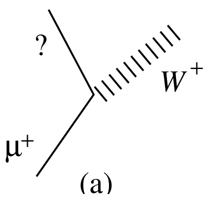
8. An antiblue antiquark emits a green-antiblue gluon. What color is the antiquark after emission?

9. Explain in your own words how *camouflage* leads to confinement of quarks inside hadrons.

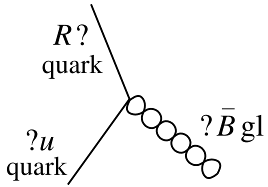
10. Draw a diagram like Fig. 10.8 for each of the reactions in Table 10.2.

11. Draw a diagram like Fig. 10.9 for each of the reactions in Table 10.3.

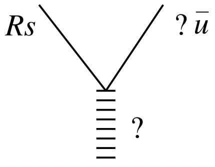
12. For each vertex diagram below, identify the missing particles, colors, or flavors denoted by a question mark. In other words, wherever you see a '?', replace it with a type of particle, a color, or a type of quark.



(a)



(b)



(c)

Chapter 11

Fundamental Forces and Interactions II

11.1 Introduction

This chapter continues from the discussion begun in the previous chapter. We now have the tools that we need to study particle interactions at the most fundamental level of quarks and leptons. In this chapter, you'll learn how to sketch particle reactions to reveal the details of the fundamental interactions. And you'll learn how to predict a particle's decay scheme and lifetime from its mass and internal structure. We'll look at the three generations of quarks and leptons and make a case for *grand unified theories* (GUTs). And finally, we'll return to the early universe as the best laboratory for testing some of these ideas.

11.2 Feynman Diagrams and Particle Interactions

We now have all the pieces that we need to produce diagrams corresponding to the particle interactions that we discussed in the previous class. All of the conservation laws that were used to analyze those interactions will be incorporated into these *interaction diagrams*.

In the previous section, we reviewed the key features of Feynman diagrams as they pertain to the various interactions. A quark or lepton goes along, then emits a messenger particle, changing the properties of the original particle. Another particle receives and absorbs the messenger, and the interaction is complete. Or perhaps the messenger decays into a quark-antiquark or lepton-antilepton pair.

We can expand the Feynman diagram idea to composite particles like hadrons by just "carrying along" the non-interacting constituents. Here's the key point: in every diagram, you should consider very carefully each vertex where a messenger is emitted or absorbed. Every vertex must conserve both charge and color. The net charge before the emission/absorption

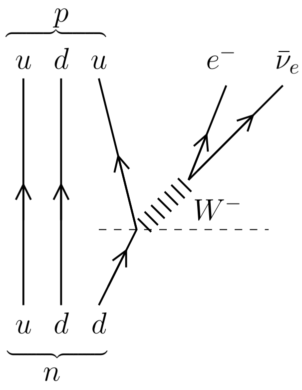


Figure 11.1: Interaction diagram for neutron decay, a weak interaction.

process is the same as the net charge after the process. Similarly, (net color before) = (net color after). If you are careful to conserve both charge and color, then your diagrams will most likely be correct.

A few examples will best illustrate this. Consider the beta-decay of a free neutron, a process governed by the weak interaction:

$$n \to p + e^- + \overline{\nu_e}. \tag{11.1}$$

To show the internal structure, simply expand the composite particle into quarks and antiquarks and let the reaction go. See Fig. 11.1. Note the two types of W-boson vertices in the diagram: one changes the flavor of a quark and the other creates a lepton-antilepton pair.

Take a look at the two vertices in this diagram. The first is the vertex corresponding to the emission of the W^- from the down quark, which is changed to an up quark by the emission. Ignore the up and down quark at the left of the diagram and consider only the particles involved at that vertex. If you draw a horizontal dotted line, then everything below the line is what was present before the emission and everything above the line is what was present after the emission. Now, consider charge conservation. Charge before: $Q_{\text{before}} = Q_d = -1/3$ (recall that Q refers to the charge of the particle). Charge after: $Q_{\text{after}} = Q_u + Q_{W^-} = +2/3 - 1 = -1/3$, so this vertex conserves charge. Note: we won't draw the horizontal dotted lines anymore.

> **Checkpoint 1:** convince yourself that charge is also conserved at the second vertex: the decay of the W^- into the e^- and $\overline{\nu}_e$.

Note that color is automatically conserved at both of these vertices, since the W particles never have any net color, nor do the e^- and $\overline{\nu}_e$.

For the strong interaction there are several possibilities. The simplest type is an *exchange reaction*, an example of which is shown in Fig. 11.2. The quarks just get jumbled up in the collision and then rearrange themselves to make up the end products. Note that there is still the exchange of a gluon

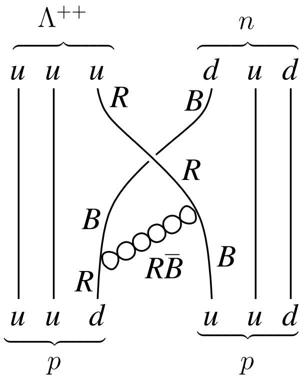


Figure 11.2: Interaction diagram for $p + p \to \Delta^{++} + n$, a strong interaction.

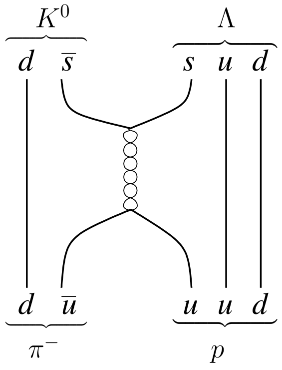


Figure 11.3: Interaction diagram for $\pi^- + p \to \Lambda + K^0$, a strong interaction.

here. Also, note that color is conserved at the vertices at both ends of the gluon. At the first (lower) vertex, the color is initially R (for the d quark in the proton). After the vertex, the color is B (for the d quark) + $R\overline{B}$ (for the gluon that has just been emitted).

> **Checkpoint 2:** convince yourself that color is also conserved at the second vertex.

Another possibility is to annihilate and create quark-antiquark pairs, mediated by a gluon. Consider the bombardment of a stationary proton with a high-energy pions, as shown in Fig. 11.3. There are several ways in which the colors could be labeled in this diagram. Let's assume that the two quarks in the π^- before the reaction have colors G and $\overline{G}$. The three quarks in the p must always have colors R, G, and B, according to the colorless rule. The gluon will then have the color combination $R\overline{G}$.

A couple of additional points to note about the reaction in Fig. 11.3. First, since strange quarks are more massive than up quarks, considerable kinetic energy (of the π^-) must be converted to mass-energy for this reaction to occur. In fact, a π^- hitting a stationary proton must have a kinetic energy of at least 770 MeV to produce a Λ and K^0. Second, the gluons that mediate the strong interaction cannot change the net *flavor* of the quarks. Up and antiup are destroyed together; strange and antistrange are created together. As a result, the additive properties like strangeness, baryon number and charge are automatically conserved.

A third possibility for the strong interaction involves the decay of *excited states* of quark configurations. This can occur whenever the mass of the parent particle is high enough to have energy available for creating a quark-antiquark pair. See Fig. 11.4 for an example.

Consider the decays of the particles labeled $\Sigma^-(1197)$ and $\Sigma^{*-}(1387)$ (the numbers in parenthesis are the masses in MeV/c^2). A proposed de-

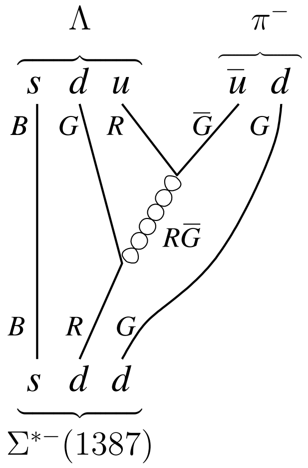


Figure 11.4: Reaction diagram for the strong decay $\Sigma^{*-} \rightarrow \Lambda(1116) + \pi^-(135)$.

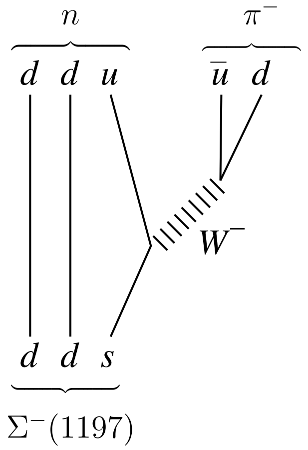


Figure 11.5: Reaction diagram for the weak decay $\Sigma^-(1197) \rightarrow n + \pi^-$.

cay scheme might be like that shown in Fig. 11.4. Note that we've shown the colors on this diagram since it involves emission and decay of a gluon. However, when we check conservation of energy, we find that the mass of the products is greater than $1197\,\text{MeV}/c^2$ but less than $1387\,\text{MeV}/c^2$. This means the $\Sigma^{*-}(1387)$ has enough rest energy to decay this way but the $\Sigma^-(1197)$ does not. It must find another way.

11.3 Particle Lifetimes

Let's see how the quark model can help us predict particle lifetimes. The two most important factors are the type of interaction involved and the amount of energy available. We'll first illustrate with an example and then develop some general rules.

In doing Problem 4, you'll discover that energy considerations severely limit the possible decay modes for $\Sigma^-(1197)$. In fact, only $\Sigma^- \rightarrow n + \pi^-$ is allowed. See Fig. 11.5. There is a flavor change, because a strange quark has to turn into an up quark, so a W^- boson must be involved, indicating the weak interaction. And because the weak interaction is much weaker than the strong, it acts much more slowly and $\Sigma^-(1197)$ lives a relatively long time. Compare the mean lifetimes: 1.5×10^{-10} s for $\Sigma^-(1197)$ and 2×10^{-23} s for $\Sigma^{*-}(1387)$. The fact that only a weak decay can occur allows $\Sigma^-(1197)$ to live 10 trillion times longer!

Here are some general rules about decay schemes and lifetimes.

- A decay proceeds by the strongest interaction allowed: strong, then electromagnetic, then weak. The typical times for these interactions are given in Tables 11.1 and 11.2.

- Particles can only decay to lighter particles. The more excess energy available, the faster the decay proceeds.
- A baryon's decay products must include a baryon.
- If quark flavor is not conserved, the decay must proceed by the weak interaction (because a W or Z boson must be involved).
- If photons are involved, the decay process proceeds by the electromagnetic interaction.
- Hadrons tend to decay to more hadrons. Only when a decay into hadrons is energetically impossible do hadrons decay to photons or leptons.

Table 11.1: Decay times for various processes.

Process	Typical Decay Time
Strong	10^{-23} s
Electromagnetic	10^{-20} s
Weak → hadrons	10^{-10} s
Weak → leptons	10^{-7} s

Table 11.2: Selected particle masses and lifetimes.

	Particle	Mass (MeV/c^2)	Lifetime	Comments
Baryons				
	p	938.3	$> 10^{30}$ yrs	
	n	939.6	898 s	
	Λ	1116	2.6×10^{-10} s	lightest $S = -1$
	Σ^0	1193	6×10^{-20} s	$\rightarrow \Lambda + \gamma$
	Σ^-	1197	1.5×10^{-10} s	$\rightarrow n + \pi^-$
	Σ^{*-}	1387	2×10^{-23} s	$\rightarrow \Lambda + \pi^-$
	Ω^-	1672	8×10^{-11} s	lightest $S = -3$
Mesons				
	$\pi^\pm$	139.6	2.6×10^{-8} s	$\rightarrow \mu + \nu$
	π^0	135	0.8×10^{-16} s	$\rightarrow \gamma + \gamma$
	$K^\pm$	494	1.2×10^{-8} s	lightest $S = \pm 1$

Table 11.3: Generations of particles

	Leptons	Quarks		
		red	blue	green
1st generation	e^-	u	u	u
	ν_e	d	d	d
2nd generation	μ^-	s	s	s
	ν_μ	c	c	c
3rd generation	τ^-	b	b	b
	ν_τ	t	t	t

11.4 Grand Unified Theories (GUTs)

Unification of the electrical and magnetic forces was explained in the late 1800's by Maxwell's comprehensive theory of electricity and magnetism, which showed that the two forces are really just different forms of the same fundamental force. More recently, the discovery of the W and Z messenger particles (in 1983) verified the *Standard Model* that unifies electromagnetic forces with weak forces (now referred to in one word as "electroweak" forces). But Einstein's dream of unification of the fundamental forces is not yet complete, since strong forces and gravitational forces have not yet been included in the model. However, attempts are being made to include the strong force with the electroweak.

To see how this inclusion would be possible, we must consider one more type of interaction, one that could change quarks into leptons and vice versa. This interaction is one of the key ideas of *grand unified theories*. But first let us reflect on the striking parallel between the set of quarks and leptons. We expand the lepton family to include quarks and call the resulting group a generation. See Table 11.3.

Only the first generation is really "needed." All the normal everyday material of nuclei, atoms, and objects are made from these. The other generations are extra pieces of the puzzle of physics. When grouped this way, one might ask if there isn't some scheme that unites the quarks and leptons. In fact there is. This is the job of grand unified theories (GUTs).

The term *grand unification* refers to the attempt to unify, in a single theory, the strong, electromagnetic, and weak interactions. That there is any hope for this to succeed comes from the fact that the strengths of the interactions seem to approach each other at short distances and thus at high energies (recall that to explore ever smaller-sized details requires more and more energetic probes). We saw in the previous chapters that camouflage effects tend to make the effective color charge (and therefore the interaction

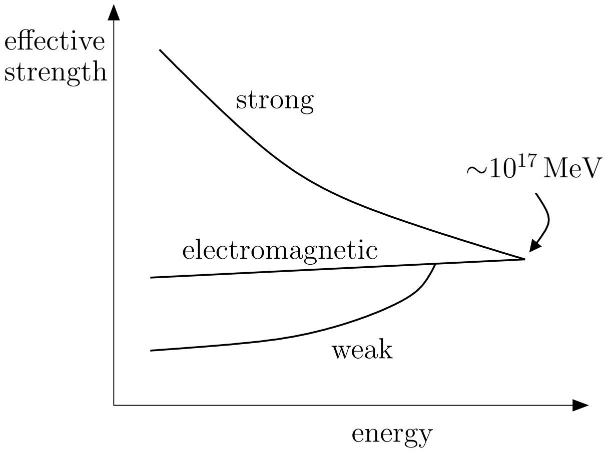


Figure 11.6: Effective strength vs. energy for strong, weak, and electromagnetic interactions.

strength of the strong force) *smaller* at closer distances. On the other hand, the weak and electromagnetic interactions get *stronger* at closer distances. In fact, the theories describing these interactions are sufficiently well understood that the distance (and thus the energy) at which the interaction strengths become comparable can be estimated to within a factor of 10. See Fig. 11.6.

The energy for grand unification, as seen from the figure, is around 10^{17} MeV. This corresponds to the rest energy of the proposed messenger particle, the X-boson or *leptoquark*. The X-boson's job is to mediate interactions in which quarks turn into leptons and vice-versa. With its tremendous mass of about 10^{14} proton masses, it could never be produced on earth as a real particle. Even in its transitory virtual state as a messenger, it would have to borrow so much energy from the surrounding vacuum, that reactions involving the X (under present-day earth-like conditions at least) are expected to be quite unlikely. However, this boson would make possible the startling reaction shown in Fig. 11.7.

If X-bosons are real, the reaction pictured — proton decay — would become possible, although extremely rare. According to present theories, the mean lifetime of the proton would be greater than 10^{30} years. Even though this is 20 orders of magnitude longer than the lifetime of the universe, it may still be testable here on earth (see Problem 3).

11.5 Universal Symmetry

One of the goals of grand unified theories is to show that the strong, weak, and electromagnetic interactions are really different aspects of a single type

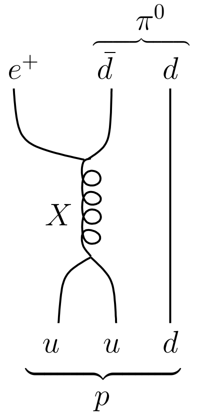


Figure 11.7: Proton decay, mediated by the X-boson.

of universal interaction. As we've just seen, by studying how the strengths of these interactions vary with energy and distance, one learns that at high enough energies, the interaction strengths would become equal (see Fig. 11.6). At this energy, one could observe directly what is known as universal symmetry. All the interactions would be on the same footing, look the same, and have the same strength.

At this grand unification energy, all interactions would be equally possible, including ones mediated by the X-boson, mentioned above in proton decay. In order to really see the X-boson in action, we would need to have energies available comparable to the X-boson's rest energy $\sim 10^{17}$ MeV. Recall that the world's biggest accelerator, the *Large Hadron Collider* in Europe, runs at a paltry 10^7 MeV. A modern linear accelerator of 10^{17} MeV would have to extend beyond the outer reaches of Pluto's orbit around the sun. So how can we hope to study such high energies where universal symmetry would be apparent?

The answer is during the early universe! Shortly after the Big Bang, the average thermal energies of particles were well in excess of the grand unification scale. In those early moments, reactions occurred that determined the character of our present universe. Since we believe that the same laws of physics governed then as now, we can use the early universe as the testing ground for modern particle physics. The details of the physics of the early universe are the subject of the next chapter.

Problems

1. Construct a reaction diagram for each of the following.

(a) $\Delta^{++} \to p + \pi^{+}$

(b) $p + e^{-} \to n + \nu_e$

2. Construct a reaction diagram for each of the following:

(a) $\pi^{-} + \Lambda \to K^{-} + n$

(b) $K^{-} \to \mu^{-} + \overline{\nu}_{\mu}$

3. Suppose the proton's mean lifetime is 10^{31} years. Now consider a pool of water $10\,\mathrm{m} \times 10\,\mathrm{m} \times 3\,\mathrm{m}$. Estimate the number of protons in the pool and then the number that you would expect to decay in one year. For this calculation, consider only the protons not bound in a complex nucleus; that is, only the protons in hydrogen atoms.

4. $\Sigma^{-}(1197)$ decays into hadrons only. List all the possible decay schemes that conserve baryon number, energy, and charge, using the tables given in Chapter 9 as a resource.

5. For each of the following reactions, tell which interaction is involved (strong, weak or electromagnetic).

(a) a green up quark emits a gluon

(b) $\gamma \to e^{+} + e^{-}$

(c) $\nu_e + n \to e^{-} + p$

6. Which lives longer, $\Sigma^{0}(1192)$ or $\Lambda(1116)$? Why?

7. An Ω^{-} particle is constructed from three s-quarks and is the lightest $S = -3$ baryon. Refer to Table 11.2 to answer the following:

(a) Why must strangeness (S) change when Ω^{-} decays?

(b) By what interaction does Ω^{-} decay?

(c) Why does Ω^{-} live relatively long?

8. The particles $\Xi^{-}(1322)$ and $\Xi^{*-}(1535)$ both carry strangeness $S = -2$. Which decays faster? Why? When the faster one decays, what are the likely decay products?

9. Pions are the lightest hadrons. What can they decay to? Why do π^{+} and π^{-} live so much longer than π^{0}?

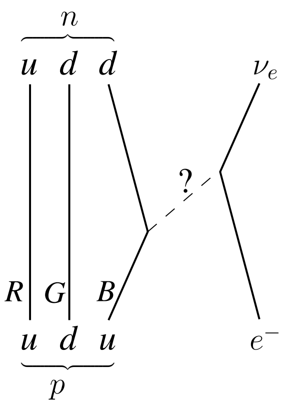
10. In Section 11.4 we discussed the possibility of proton decay. Fig. 11.7 shows a *hypothetical* process in which proton decay is mediated by a *proposed* $X^{4/3}$ boson, which carries a $+\frac{4}{3}$ charge. (The $X^{4/3}$ boson has not been observed, and is not one of the particles included in the tables in this supplement.) Note that the general rule for interactions involving an X-boson would be:

$$\text{quark} + \text{quark} \to X \to \text{antiquark} + \text{antilepton}$$

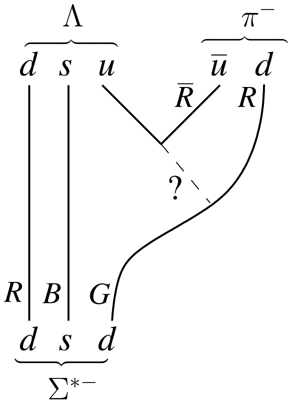
Another possible proton decay scheme, using a $X^{1/3}$ boson (with $+\frac{1}{3}$ charge), would be $p \to \pi^+ + \overline{\nu}_e$. Construct a reaction diagram like Fig. 11.7 showing this hypothesized decay process. Be sure to conserve charges at each vertex!

11. In Fig. 11.1, label the colors of any particle in the diagram that has a color. Be sure to satisfy the *colorless rule* for any hadron, and don't label colors on any particle that does *not* carry color. (Note: there are several right answers to this question.)

12. For each of the following reaction diagrams, identify the messenger particle and state the type of interaction involved (strong, weak, electromagnetic, or gravitational). Denote the colors and/or charges for each messenger, where appropriate.



(a)



(b)

Chapter 12

Cosmology

12.1 Introduction

Cosmology is the study of the origins of the cosmos, that is, the history of the universe. The present standard model of cosmology is the subject of this chapter.

At the end of Chapter 11, we claimed that the temperature and available energy in the early universe were sufficient to achieve universal symmetry. In this chapter we present a simple relation that shows how energy and temperature decrease with time. We then look in detail at each of the major events that served to mark transitions from one phase of the universe to another. We'll learn why these events occurred when they did, and what ramifications these events had for the interactions that followed. And finally we'll see what observational evidence exists that could test the microscopic theories of particle physics.

12.2 The Early Universe

The standard theory of cosmology is the Big Bang model, in which the universe began as a hot dense ball of stuff and has been expanding and cooling ever since. The first and primary evidence for this model was Edwin Hubble's discovery in the 1920's that the universe is expanding. From analysis of Doppler shifted spectra and other observations, Hubble showed that most distant galaxies are receding from our galaxy. Furthermore, their velocity of recession is roughly proportional to their distance from us.

To see how this could happen, imagine baking a giant loaf of raisin bread. As the dough expands, each raisin gets farther from all the others and, if the expansion is uniform, the recessional rates at any instant of time are proportional to the separations. The recession rates of galaxies in the expanding universe are similarly proportional to the separation between galaxies. The proportionality can be expressed simply as

$$V = HR \qquad \text{(Hubble's Relation)} \tag{12.1}$$

where V is the velocity of recession, R is the galaxy distance and H is Hubble's constant. By constant, we mean it is the same number for many different galaxies, not that it doesn't change with time. In fact we argue below that it must be decreasing with time. Its current accepted value is 20.8 km/s per million light-years. This means that if galaxy B is a million light years more distant than galaxy A, then B is receding 20.4 km/s faster than A.

Okay, now here is the key point: if the universe is expanding, then it is also cooling down. Especially when considering the early moments of the universe, you can compare it to a gas that is undergoing an adiabatic expansion (no heat flow into or out of the system, which is the entire universe in this case). Recall from PHYS 211 that a gas cools down when undergoing an adiabatic expansion. It can be shown that[1] the temperature T of the universe during its early stages — during the "radiation-dominated" epochs, i.e., the first million years — is given roughly by the equation

$$T = \frac{10^{10}}{\sqrt{t}}, \tag{12.2}$$

where time t is in seconds and temperature T is in Kelvin. Even though the assumptions of this derivation[2] are very much oversimplified, Eq. (12.2) gives results correct within a factor of 2 or 3 up to $t = 10^6$ years. After that time, the universe is cool enough to start acting like a collection of material particles rather than a bunch of photons.[3] Thus, for times near the present, Eq. (12.2) overestimates the temperature of radiation considerably.

A useful relation can be obtained from Eq. (12.2). Recall from kinetic theory (PHYS 211) that for particles in equilibrium, the average kinetic energy equals $\frac{3}{2}kT$, where Boltzmann's constant $k = 8.63 \times 10^{-5}\,\text{eV/K}$. If we round these figures to the same accuracy as Eq. (12.2), we find $E \simeq kT$ with $k \simeq 10^{-4}\,\text{eV/K}$, and thus typical particle energies of the early universe are given by

$$E \simeq kT \simeq \frac{10^6}{\sqrt{t}}, \tag{12.3}$$

where energy E is in eV and time t is again in seconds. Equations (12.2) and (12.3) are handy relations that will be used throughout the chapter.

---

[1] "It can be shown that" is physics-lingo for "the derivation is a royal pain-in-the-neck, so we will just skip it here."

[2] Which we haven't told you.

[3] One of the assumptions that we haven't explained.

Table 12.1: Events of the early universe.

Time	Event
0	Big Bang
10^{-43} s	Planck Time — end of Universal Symmetry
10^{-34} s	End of Grand Unification
10^{-10} s	End of Electro-weak Unification
10^{-4} s	Confinement of quarks into hadrons
1 s	Neutrinos stop interacting (decouple)
400 s	Protons and neutrons combine to form nuclei
10^{5} yr	Nuclei and electrons combine to form atoms

12.3 Crucial Events of the Early Universe

The universe began some 14 billion years ago with a Big Bang and has been expanding and cooling ever since. Table 12.1 is a list of the crucial events that occurred early on, most of which we will try to explain.

How can we possibly determine when these events occurred? Except for the end of neutrino interactions and the time of quark confinement, each of these events can be understood in terms of the energies available to maintain thermal equilibrium. For instance, the end of the electro-weak unification occurred when the massive gauge bosons of electro-weak theory (the W^+, W^-, and Z^0) could no longer be produced in prodigious amounts from random collisions of particles in thermal equilibrium. In other words, as the universe cooled, the ambient thermal energy became insufficient to create such massive particles. When this occurred, the large mass of the W and Z differentiated them from the massless photon, the messenger particle of electromagnetic theory. After this time the weak and electromagnetic interactions were observed to behave differently.

So how can we relate the time of the splitting off of the weak interaction to properties of the W's and Z's? Here's how. The theoretical model for electro-weak unification, the Weinberg-Salam theory of 1973, predicted rest energies for W and Z in the range of 80–100 GeV. This prediction was dramatically confirmed when the rest energies of W (84 GeV) and Z (92 GeV) were measured in the process of their experimental discovery at CERN in 1983. If we round these rest energies up to $E = 100\,\text{GeV} = 10^{11}\,\text{eV}$, we can use Eq. (12.3) to estimate the time when this much energy was available as ambient thermal energy. We find

$$t = \left(\frac{10^6}{E}\right)^2 = \left(\frac{10^6}{10^{11}}\right)^2 = 10^{-10}\,\text{s} \tag{12.4}$$

as shown in the table. In general, the energy available at the time of all but two of the events above is related either to the rest energy of a particle or to a reaction energy associated with the event. The problems expand on this relationship.

Finally, let's try to explain the remaining two events: the decoupling of neutrinos from thermal equilibrium with the rest of the universe and the confinement of quarks inside hadrons. Both of these events depend on the average density of the universe which, as the universe expands, is continually decreasing.

Using some of the assumptions needed (but not presented here) to derive Eqs. (12.2) and (12.3), along with estimates of the present average density of the universe, we find a density of $4 \times 10^5\,\mathrm{g/cm^3}$ (and thus 4×10^5 times that of water) for neutrino decoupling and $4 \times 10^{13}\,\mathrm{g/cm^3}$ for quark confinement. Now neutrinos are very penetrating particles. Those produced in modern accelerators can penetrate miles of plate steel with less than one in a million being absorbed! The high energy neutrinos of the early universe were even more penetrating. When the universe's density fell to $4 \times 10^5\,\mathrm{g/cm^3}$, the absorption rate of neutrinos (in reactions like $\nu + p \rightarrow n + e^+$) fell significantly below their production rate, and neutrinos could no longer exchange enough energy with the rest of the universe to maintain thermal equilibrium.

Quark confinement is explained by density considerations in a similar way. Before $10^{-4}\,\mathrm{s}$, individual quarks were "free" in the sense that they did not travel around through space in groups of quark-quark-quark (baryons) or quark-antiquark (mesons). A single colored quark could go where it pleased. But this was only because the density of the universe was comparable to the density of an atomic nucleus, around $10^{14}\,\mathrm{g/cm^3}$. The entire universe was one big nucleus! As a result, any individual quark was never far from lots of other quarks, and the gluon "rubber band" never had to be stretched to give rise to confining forces. After this time, the density decreased enough that spaces started forming between nucleons and other particles, and quarks became confined inside hadrons in colorless combinations.

12.4 Epochs of the Early Universe

Let's now see what was going on in the universe during each epoch, the intervals between the crucial events of Section 12.3, and explain briefly why.

Bang to 10^{-43} s: Very speculative. This would be the time of universal symmetry, with quantized gravity incorporated with the other three interactions. Nobody really knows how.

10^{-43} to 10^{-34} s: Grand unification prevails. The strong, weak and electromagnetic interactions are on equal footings. Quarks and leptons are not yet distinct, since there are plenty of X bosons to facilitate interconversion.

10^{-34} to 10^{-10} s: The strong interaction acts separately from the electroweak. The X bosons are now too heavy to be produced in random collisions and they decay away quickly leaving distinct quarks, antiquarks, leptons and antileptons. However, due to a very subtle flaw in symmetry (which we won't explain here), the X boson and its antiparticle ($\overline{X}$) decay slightly differently. This leaves a slight excess (about one part in 10^9) of quarks over antiquarks. The universe now consists of a hot "soup" of quarks, leptons, and gauge bosons (W's and Z's, gluons, photons) all in exact thermal equilibrium.

10^{-10} to 10^{-4} s: The weak interaction now acts separately from electromagnetism, because W's and Z's are no longer produced abundantly. By the end of this epoch, the density of the universe has fallen sufficiently that quarks begin collecting into hadrons.

10^{-4} to 1 s: With quarks permanently confined, the "soup" now consists of hadrons and leptons. Energies are now insufficient to create exotic particles or to produce hadronic particle-antiparticle pairs. Thus annihilation of antiparticles occurs rapidly and essentially all baryonic antimatter is wiped out, leaving the small original (since the time $t = 10^{-34}$ s) excess of matter. We are left with primarily protons, neutrons, electrons, positrons, neutrinos and photons in thermal equilibrium. So much radiation is released from the annihilation of quarks that there are approximately 10^9 photons/baryon, the value we observe today.

1 to 400 s: The universe is now diffuse enough ("only" about 400,000 times as dense as water) that neutrinos don't interact enough to stay in equilibrium with the rest. From 1 second on they evolve separately. This means that inverse beta decay:

$$\overline{\nu}_e + p \to n + e^+ \tag{12.5}$$

stops turning protons back into neutrons. But a neutron can spontaneously decay to a proton, with 10 percent decaying every 100 seconds. So the ratio of protons to neutrons starts growing.

Also during this epoch the available thermal energy falls below that required to create electron-positron pairs and so positrons pretty much disappear. At the same time, thermal energies start to fall below the binding energies of small nuclei, most notably the deuteron. This means that protons and neutrons can start coalescing to form helium and lithium nuclei without being blasted apart by high energy photons. By the end of this period, essentially all the neutrons are bound into light nuclei. The ratio of protons to neutrons and the relative abundances of primordial elements are fixed at this time.

400 s to 10^5 years: The universe consists of a plasma of protons, helium nuclei, electrons and photons, and decoupled neutrinos. Any remaining free neutrons have long since decayed away. The photons interact easily with the free electrons via the Compton effect and thermal equilibrium between radiation and matter is maintained.

10^5 years to the present: At last it is cool enough that electrons and nuclei can combine to form atoms, since the photon energies are too low to easily ionize the atoms. Later the photon energies fall below even that level necessary to excite atomic transitions, so there is no longer any efficient way for photons to transfer energy. The universe becomes transparent to light as the photons decouple from the matter. Those primordial photons are still wandering around the universe.

12.5 Observational Tests

Three pieces of observational evidence can serve as a testing ground for the theories of what went on in the early universe:

- the discovery of the 3 K microwave background radiation,
- the measurement of the ratio of photons to baryons,
- the observation of the relative abundances of hydrogen, helium, deuterium, and lithium in the interstellar medium.

Let's look at each in turn.

Penzias and Wilson's discovery of the 3 K microwave radiation in 1965 was a stunning confirmation of the Big Bang model. This radiation seems to be coming from everywhere in the universe. It has a *blackbody* spectrum corresponding to a body at 3 K. A simple calculation shows that these photons are the remnants of the Big Bang, the photons that decoupled at $\sim 10^5$ years and have been cooling ever since!

The present observed ratio of photons to baryons seems to be about 10^9. Recall our previous claim that these photons are those produced during the annihilation of almost equal numbers of hadronic particles and antiparticles. If Grand Unified Theories (GUTs) are correct in their description of the X-boson, then the slightly different decay modes for $\overline{X}$ and X should lead to a quark excess (over antiquarks) of about 1 in 10^9. Various versions of GUTs are now being developed, and each version will presumably lead to an estimate of the X-$\overline{X}$ decay asymmetry. See Problem 9. The 1 in 10^9 figure thus provides an "experimental" test of various possible grand unified theories. It is known so far that the simplest conceivable scheme for *grand unification* (so-called minimal coupling) *does not* get the X and $\overline{X}$ decay modes right, and thus is effectively ruled out as a viable theory on observational grounds.

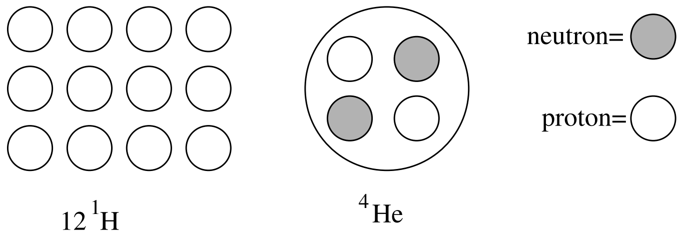


Figure 12.1: Fourteen protons and two neutrons combine to form one ^{4}He nucleus, with 12 protons left over.

Finally let's consider the question of relative abundances of the light elements. The more neutrons there were at the end of the time of nuclear synthesis, the higher the ratio of helium to hydrogen would be. (Ignore for the moment all the other elements; these constitute less than 1% of the mass of the universe.) For example, if there were seven times as many protons as neutrons, then in a group of sixteen baryons, fourteen would be protons and two would be neutrons. After combination (see sketch below), there is one helium nucleus (mass 4) and twelve hydrogen nuclei (total mass 12). The universe would then be 25% helium, by weight. See Fig. 12.1.

One of the things the exact proton to neutron ratio depends on is the number of different types of neutrinos that exist. The more types there are, the longer the neutrinos could initiate proton-to-neutron conversions (inverse beta decay) in the period just before nuclear synthesis, and the more neutrons there would be. This in turn would lead to a higher helium abundance in the present universe. Using standard nuclear physics, the helium abundances for two, three, or four neutrino types can be calculated and compared to the observational abundance data. It turns out that three neutrino types is the best fit. This means that cosmology can tell us what no theoretical arguments have yet been able to show, that there are just three neutrino types and thus just three generations of quarks and leptons! The predictions have been confirmed recently by measuring the uncertainty in the mass of the Z^0 boson.

Problems

1. Use the time in Table 12.1 to calculate the rest energy associated with the end of Grand Unification. What particles would have about this mass?

2. About how long after the Big Bang would there be insufficient energy to create Λ-$\overline{\Lambda}$ pairs? The rest energy of a Λ is 1116 MeV.

3. Assuming that one photon is released in each baryon-antibaryon annihilation, why are there 10^9 photons/baryon after annihilation and not some other number?

4. What is the thermal energy per particle available at $t = 1\,\mathrm{s}$? How does this compare with the combined rest energies of an electron-positron pair? What happens to positrons after this time?

5. List the types of particles you might expect to find abundantly in the early universe during the following epochs:

(a) 10^{-10} s to 10^{-4} s after the Big Bang

(b) 1 s to 400 s after the Big Bang

(c) 10^5 years after the Big Bang

6. What particle energy is associated with the time of quark confinement? What hadron has about this rest energy?

7. Use Eq. (12.2) to estimate the photon temperature at the present time (∼10–15 billion years after the Bang). Note that this is an overestimate since the assumptions of Eq. (12.2) have broken down.

8. Transition energies for typical atomic states are about 1–10 eV. That is, photons in this energy range are captured by atoms to excite upward transitions. Use Eq. (12.3) to estimate when in the early universe this "capture effect" became energetically impossible.

9. The X and $\overline{X}$ bosons decay slightly differently because of a small flaw in charge conjugation invariance. Suppose the decay scheme for X had the following branching ratios:

$$\begin{tikzcd}
 & q+q \\
X^{4/3} \arrow[ur, "90\%"] \arrow[dr, "10\%"'] & \\
 & \overline{q}+\overline{l}
\end{tikzcd}$$

That is, there is a 90% probability that X decays to two quarks, and a 10% chance that X decays to an antiquark plus an antilepton. Similarly suppose $\overline{X}$ decays by

$$\begin{tikzcd}
 & \overline{q}+\overline{q} \\
\overline{X}^{-4/3} \arrow[ur, "89\%"] \arrow[dr, "11\%"'] & \\
 & q+l
\end{tikzcd}$$

(a) Take 3000 random X's and 3000 random $\overline{X}$'s. How many each of quarks, leptons, antiquarks, and antileptons would appear after all X's and $\overline{X}$'s decayed?

(b) Let all the quarks combine into baryons, and all the antiquarks combine into antibaryons. How many of each are there?

(c) Now let all the particles and antiparticles annihilate as completely as possible. What's left over? Assume one photon is released from each annihilation.

(d) Calculate the photon to baryon ratio for this $(X, \overline{X})$ decay scheme.

Answers to Selected Problems

Additional Problems

A6 (a) $-5560\,\hat{\jmath}\,\mathrm{N/C}$; (b) 0; (c) $5560\,\hat{\jmath}\,\mathrm{N/C}$.
A7 (a) $k\lambda/a$; (b) $-k\lambda/a$.
A8 $\frac{-2qk}{\pi R^2}(\hat{\imath}+\hat{\jmath})$.
A9 (a) $50,000\,\mathrm{eV}$; (b) $8.0\times10^{-15}\,\mathrm{J}$; (c) $1.33\times10^{8}\,\mathrm{m/s}$.
A11 left side: $1600\,\mathrm{V/m}$ to left; right side: $300\,\mathrm{V/m}$ to right.
A18 $v_f=\sqrt{2IHBD/m}$.
A20 $4\times10^{-5}\,\mathrm{T}$.
A22 $5.8\times10^{-3}\,\mathrm{T}$.
A23 $0.013\,\mathrm{s}$.
A24 $\pi r^2 B_0(6t-1)/R$.
A31 (a) $\pm z$, (b) $\pm x$, (c) $\pm z$, (d) $\pm z$, (e) $\pm x$, $\pm y$.
A36 (a) $20\,\mathrm{m}$, $10\,\mathrm{m}$, $6.67\,\mathrm{m}$; (b) $\pi/10$, $\pi/5$, $3\pi/10$; (c) $17\,\mathrm{Hz}$, $34\,\mathrm{Hz}$, $51\,\mathrm{Hz}$.
A45 (a) $717\,\mathrm{nm}$; (b) $55,800$.
A48 (a) $\lambda=1257\,\mathrm{nm}$, $T=4.19\times10^{-15}\,\mathrm{s}$, $v=3.0\times10^{8}\,\mathrm{m/s}$, infrared, (b) $B=0.2\,\mathrm{mT}$;
A58 (b) $2\times10^{-14}\,\mathrm{m}$; (c) $p=3.3\times10^{-20}\,\mathrm{kg{\cdot}m/s}$, $K=.3.3\times10^{-13}\,\mathrm{J}$.
A62 $1.3\times10^{-21}\,\mathrm{m/s}$.
A66 (a) $\Delta t=6.6\times10^{-18}\,\mathrm{s}$; (b) $v=4.2\times10^{6}\,\mathrm{m/s}$; (c) $L=2.8\times10^{-11}\,\mathrm{m}$; (d) 0.13.
A79 $5.4\,\mathrm{cm}$.
A80 $19.4\,\mathrm{mT}$.
A87 $1.41A$.
A89 (a) minimum; (b) $9I_0$.
A91 $y(x,t)=2\cos(\pi x-2\pi t)$.
A93 (b) 2.95; (c) $0.5\,\mathrm{rad}$.
A95 (a) $\frac{1}{1600\pi}\,\mathrm{m}^{-2}$; (b) 1/4; (c) $P_A=\frac{1}{1000\pi}\,\mathrm{m}^{-2}$, $P_B=\frac{1}{2000\pi}\,\mathrm{m}^{-2}$; (d) 13/20.
A103 (b) 2.
A106 (a) e^+, W^+; (b) e^+, e^+; (c) Ru, Gd.
A110 $4.3\times10^{-6}\,\mathrm{T}$.
A111 (a) $0.1\,\mathrm{mT}$; (b) $0.2\,\mathrm{mT}$; (c) $0.1\,\mathrm{mT}$.

Chapter 1
2. (b) $x(t) = 30\cos(\frac{\pi}{3}t + \pi/2)\,\text{cm}$.
5. $3\sqrt{5}$.
7. $1.00A$.
9. (b) $\theta = 0.45°$, $0.90°$, $1.35°$.

Chapter 2
1. $E_{ph} = 2.48\,\text{eV}$, $K = 6.0 \times 10^{-6}\,\text{eV}$.
3. (a) $f_c = 5.4 \times 10^{14}\,\text{Hz}$.
4. (a) 10^{-9} m or smaller, $E_{ph} = 1240\,\text{eV}$; (b) 10^{-9} m or smaller, $K_{electron} = 1.5\,\text{eV}$.
5. $p = 4140\,\text{eV/c}$ or 2.21×10^{-24} kg m/s.
6. $1.1\,\text{eV}$.
7. (a) $E_{ph} = 2.48\times10^{-13}\,\text{eV}$; (b) $E_{ph} \ll U_{bind}$, so no chemical bonds affected, including those in DNA.
10. (a) $\frac{hc}{2L}j$; (b) $\frac{Lk_BT}{hc}\frac{1}{j}$; (c) $j_{max} = \frac{Lk_BT}{hc}$.
13. $K_{max} = 7.1\,\text{eV}$.

Chapter 3
4. 0.42.
5. (a) 1.06×10^{-24} kg m/s; (b) 1.15×10^{6} m/s.
6. 2.1×10^{-26} m/s.
7. (a) 6.6×10^{-13} m; (b) 2.3×10^{-12} m; (c) (Hint: do you think the binding energy for a proton in a nucleus is 1 eV?)
10. (a) 6.1×10^{-19} J or $3.8\,\text{eV}$; (b) 1.5×10^{-33} J or 9.6×10^{-15} eV; (c) 6.9×10^{-61} J or 4.3×10^{-42} eV.
13. (a) Works if $A = 5/6$ and $B = 3$; (b) Doesn't work for all values of x.
14. (b) Doesn't work; (c) Works if $k = \pm\sqrt{2m(E-U_0)}/\hbar$, which is fine since $E > U_0$; (d) Would work if $\kappa = \pm\sqrt{2m(U_0-E)}/\hbar$, but $E > U_0$, so this would be an imaginary κ.
16. $E = 2$ for a solution.

Chapter 4
1. (b) $2L/3$; (c) $E = \frac{9h^2}{8mL^2}$.
2. (a) $0.38\,\text{eV}$; (b) $1.51\,\text{eV}$.
4. $L/4$ and $3L/4$.
10. (a) $7.2 \times 10^{9}\text{m}^{-1}$; (b) 4.8×10^{-11} m; (c) 5.2×10^{-36} m.
12. $47.2\,\mu$m, $27.5\,\mu$m, $22.0\,\mu$m, $66.0\,\mu$m, $41.2\,\mu$m, $110\,\mu$m.
13. 5.18 nm.
14. (a) 315 nm; (b) 240 nm.
16. $15\,\mu$m and $7.5\,\mu$m.

Chapter 5
3. (a) $\frac{1}{2}$; (b) 0; (c) $\frac{1}{2}$.
5. (a) $\frac{1}{5}$; (b) $\frac{1}{10}$; (c) $\frac{1}{2}$; (d) 0.

6. (a) $\frac{1}{2}$; (b) $\frac{1}{2}$.
7. (b) $\cos^2\theta$, $|\mathrm{X}\rangle$; (c) $\cos^2\theta$, $|\theta\rangle$; (d) $\sin^2\theta$; (e) $\cos^2\theta\sin^2\theta$.
10. (a) 0.75; (b) 0.067.
11. 0.50.
14. 21 cm.
17. (a) $\frac{13}{36}$; (b) 0; (c) There are numerous answers to this question. The easiest are $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$ or $i\frac{\sqrt{2}}{2}$ or $-i\frac{\sqrt{2}}{2}$. There are also numerous $a+ib$ combinations that would work, as long as $|a+ib|^2=\frac{1}{2}$.
19. (a) Two equal intensity beams (1/4 the intensity of the intensity of the initial electron beam) will emerge, one with $S_z=+\hbar/2$ and the other with $S_z=-\hbar/2$. (b) A beam will emerge from just one of the two exits with $S_x=-\hbar/2$. This beam will be 1/2 the intensity of the initial electron beam.

Chapter 6

1. (a) electrons are indistinguishable, (b) $\frac{1}{\sqrt{2}}|E_1\uparrow\ E_1\downarrow\rangle-\frac{1}{\sqrt{2}}|E_1\downarrow\ E_1\uparrow\rangle$.
2. Yes for both (a) and (b). Electrons and muons are distinguishable.
3. (b) 4 for classical, 3 for bosons, 1 for fermions; (c) $\frac{1}{2}$ for classical, $\frac{2}{3}$ for bosons.
4. 84 eV.
8. ^{3}He is a fermion.
10. 0 (that's the point!).

Chapter 7

3. 6.84×10^{-34} J·s.
4. -1.51 eV.
7. (a) 2.8×10^{-18}, 0.017 and 0.40, (b) 0.026 eV, 0.26 eV and 2.6 eV; (c) 0, 0, and 7.
9. Would expect poor conductivity at 300 K, moderate conductivity at 3,000 K, and excellent conductivity at 30,000 K.
11. (a) 867 nm (near-IR); (b) 549 nm (yellowish green); (c) 365 nm (near-UV).
15. (b)$E=-\frac{mk^2e^4}{2\hbar^2}$.

Chapter 8

1. (a) 3/10, (b) 7/10.
3. (a) $c_+=\sqrt{3/10}$, $c_-=\sqrt{7/10}$, $|\phi_1\rangle=\sqrt{1/3}\,|\uparrow\rangle+\sqrt{2/3}\,|\downarrow\rangle$, $|\phi_2\rangle=\sqrt{2/7}\,|\uparrow\rangle+\sqrt{5/7}\,|\downarrow\rangle$.
4. (a) 0.55; (b) $|\psi_{\text{new}}\rangle=1.0\,|\uparrow\rangle\,|\phi_1\rangle$; (c) $\frac{1}{3}$.
7. 0.854.
9. (a) 0.75, (b) 0.67.

Chapter 9

2. Mesons are bosons, baryons are fermions.
6. (a) K^+, (b) K^+.

7. (a) yes, (b) no, (c) no, (d) yes.
10. Does not conserve charge.
11. $S = -2$.
12. (a) uds, (b) dss, (c) $u\bar{d}$.
13. This requires two s-quarks, with total charge $-2/3$. No single quark can add $q = 5/3$.
14. (a) n or Δ^0, (b) Σ^+ or Σ^{*+}, (c) K^0.

Chapter 10

2. (a) 1.022 MeV, (b) 6.5×10^{-22} s.
3. (a) 91.2 GeV, (b) 7×10^{-27} s, (c) 2×10^{-18} m.
8. antigreen.

Chapter 11

3. About two protons should decay.
4. Only $\Sigma^- \to n + \pi^-$.
5. (a) strong, (b) electromagnetic, (c) weak.
7. (a) There are no lighter baryons with $S = -3$, (b) weak; strangeness is not conserved, (c) weak interaction is slower than strong or electromagnetic.
8. $\Xi^-(1535)$ decays much faster; it can decay by the strong interaction; $\Xi^-(1535) \to \Xi^-(1322) + \pi^0$, while the lighter Ξ^- must go by a weak interaction.
9. Photons or leptons. Because the weak decay to leptons is much slower than the electromagnetic decay to photons.

Chapter 12

2. about 10^{-7} to 10^{-6} s
6. about 100 MeV, pions
9. a) 5730 q, 5640 $\bar{q}$, 330 l, 300 $\bar{l}$, b) 1910 baryons, 1880 antibaryons, c) 30 baryons, 30 leptons, 2180 photons, d) 72.7