PHYSICS 222 WAVE MECHANICS AND MODERN PHYSICS
Problem Set 13
SPRING, 2017
J.J. Thomson, Rutherford, and Bohr: the Early Atomic Era

Problem 1
Using J.J. Thomson's model for the hydrogen atom, the electron experiences simple harmonic motion due to its electric attraction to a ball of 'positive paste.'

(a) Compute the oscillation frequency, assuming a typical hydrogen atom radius of 0.53 Å.

(b) Determine the wavelength of the electromagnetic radiation emitted by this accelerating charge and compare it with the observed wavelength of the strongest emission/absorption line in hydrogen, 1.22 nm. Comment on your observations.

Problem 2
Work through the derivation of Thomson's calculation for the scattering angle expected for a projectile of mass m and charge ze, striking an atom of charge Ze.

(a) By calculating the off-axis impact (Δp_y), show that the scattering angle is given by

$$\tan\theta \approx \theta = \frac{2zKb}{mv^2}\sqrt{R^2 - b^2},$$

where $K = kZe^2/R^3$ and k is Coulomb's constant, $1/(4\pi\varepsilon_0)$.
Assume an initial projectile speed of v, an impact parameter b, an atomic radius R, and that the impact occurs during the interval of time that the projectile is travelling through the atom.

(b) Determine the maximum scattering angle for the Thomson atom, along with the impact parameter for which it occurs.

Problem 3
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 42.
NB: In contrast to the previous question, this problem has you applying Rutherford's scattering formula.

Problem 4 (assigned)
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 10.
Compare your result with the actual energy of alpha particles used by Geiger and Marsden. Comment on your comparison.

Problem 5 (assigned)
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 48.
This provides good practise using Rutherford's theoretical results to make a prediction for a measurement that can be tested experimentally.

Problem 6
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 24.

Problem 7
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 43.

Task 8 (assigned)
Read through Shamos' chapter "The Hydrogen Atom" on the work on Niels Bohr in his book *Great Experiments in Physics*. Paying particular attention to §1 of Bohr's paper, write a paragraph comparing and contrasting his presentation of the 'Bohr model' to that presented in our textbook, Tipler & Llewellyn.

Problem 9
The Correspondence Principle.

(a) State Bohr's postulates and derive Bohr's expression for the n^{th} energy level of a hydrogen-like atom having nuclear charge, $+Ze$.

(b) State the Correspondence Principle.

According to the correspondence principle, when energy levels are closely spaced, quantisation should have little effect. Using Bohr's theory of the atom, closely spaced energy levels occur when the quantum number, n, is large.

We will investigate the correspondence principle in relation to the frequency of a photon of light emitted by an atomic transition. Consider the photon emitted as a result of a transition of the electron from the $n_i = n$ to the $n_f = n - 1$ energy level.

(c) Using the results of Bohr's theory for an atom of nuclear charge Ze, show that the frequency of the emitted photon is given by

$$f_{qu} = \frac{Z^2 mk^2 e^4}{4\pi\hbar^3} \frac{2n-1}{n^2(n-1)^2}.$$

(d) Show that this reduces to

$$f_{qu} \approx \frac{Z^2 mk^2 e^4}{2\pi\hbar^3 n^3}$$

for large n.

(e) Classically, electromagnetic theory predicts that an electron 'orbiting' the nucleus will emit radiation at the same frequency as the frequency of motion of the moving charge, i.e. the electron.

Show that the classical frequency of revolution of the electron in the E_n energy level is given by

$$f_{cl} = \frac{mk^2 Z^2 e^4}{2\pi\hbar^3 n^3}$$

which is identical to the frequency using Bohr's quantum approach.

Problem 10
Tipler & Llewellyn: *Modern Physics* Chapter 4, problem 45.

Problem 11
A hypothetical atom has only two excited states, at 4.0 and 7.0 eV, and has a ground state ionisation energy of 9.0 eV. If we used a vapour of such atoms for the Franck-Hertz experiment, determine the voltages at which we would expect to see decreases in the current. List all voltages up to 20 V, and make a sketch of the current as a function of voltage.

T&L 4-42

Geiger and Marsden used α-particles with $K = 7.7\,\text{MeV}$.
This arrangement produced scattering events that were consistent with Rutherford's prediction of scattering from a positive point charge, $+Ze$.

To determine the upper limit on the Au nucleus radius:

for Au, $Z = 79$

consider a head-on collision,

before:

r_{cl} : distance of closest approach

Applying conservation of energy

$$\left(K + \underbrace{U}_{0}\right)_{before} = \left(\underbrace{K}_{0} + U\right)_{final}$$

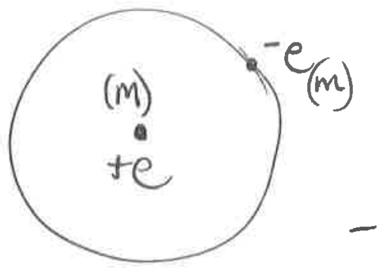
$$\frac{1}{2} M_\alpha v_\alpha^2 = \frac{k(+2e)(+79e)}{r_{cl}}$$

$$\therefore\ r_{cl} = \frac{k \cdot 2 \cdot 79 \cdot e^2}{7.7 \times 10^6 \cdot e} = 2.95 \times 10^{-14}\,\text{m}$$

$\therefore$ radius of Au nucleus is at **most** $29.5\,\text{fm}$.

T. & L. 4-24

Positronium is a bound state of an electron-positron pair



Since e^+ & e^- have the same mass, circular motion occurs about the centre-of-mass - which is at the midpoint between the two particles

In this case, the reduced mass of the system plays a significant role:

Using the Bohr model,
we have $E_n = \frac{E_1}{n^2}$ where $E_1 = hc \cdot R$ ← Rydberg constant

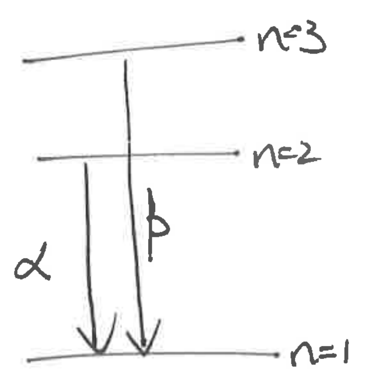
and $R = R_\infty \left(\frac{1}{1 + m/M} \right)$
↑ R when $M \ggg m$

$R_\infty = 1.0973732 \times 10^{-2}\ nm^{-1}$

For the case of positronium,
$\frac{m}{M} = 1 \quad \therefore R = \frac{1}{2} R_\infty$

$\therefore E_n = hc \frac{1}{2} R_\infty / n^2$
$= \frac{1}{2}(-13.6)\,eV / n^2$

(a)
$\therefore$ for $n=1$, $E_1 = -6.80\ eV$
$n=2$, $E_2 = -6.80/2^2 = -1.70\ eV$
$n=3$, $E_3 = -6.80/3^2 = -0.76\ eV$



(b) Lyman series ⇒ drops to n=1 level
$E_{ph} = E_n - E_1 = \frac{hc}{\lambda}$

$\lambda_{Lyman\ \alpha} = 243\ nm \quad n=2 \to n=1$
$\lambda_{Lyman\ \beta} = 205\ nm \quad n=3 \to n=1$

T. &L. 4-43

(a) Consider a charge, q, moving in a circular orbit of radius, r.

$i = \frac{\text{charge}}{\text{time}} = \frac{q}{T}$ where T = period

$\therefore \; i = q f_{rev}$ since $f_{rev} = \frac{1}{T}$.

For an electron in the first Bohr orbit

$n = 1 \Rightarrow r = a_0 = 0.53\,\text{Å}$

also $L = 1\hbar$

now $\vec{L} = \vec{r} \times \vec{p}$

$\therefore \; L = rmv$ so $rmv = \hbar$

$\therefore \; v = \frac{\hbar}{r.m}$

also $v = \frac{2\pi r}{T} = 2\pi r f$

$\therefore \; f = \frac{v}{2\pi r}$ so, $f = \frac{\hbar}{2\pi r^2 m}$

$\therefore \; i = \frac{e.\hbar}{2\pi r^2 m} = \frac{(1.6\times10^{-19})(1.05\times10^{-34})}{2\pi(0.53\times10^{-10})^2.(9.1\times10^{-31})}$

$\therefore \; i = 1.05\times10^{-3}$ Amp.

(b) $\mu = i.A. = i.(\pi r^2)$

$\therefore \; \mu = (1.05\times10^{-3})(\pi.(0.53\times10^{-10})^2)$

$\therefore \; \mu = 9.27\times10^{-24}\ \text{A}.\text{m}^2$: the Bohr magneton.

Bohr's postulates:

1. electrons could move in certain, allowable orbits without radiating – called stationary states

2. an atom radiates when an electron makes a transition from one stationary state to another, and the frequency of emission corresponds to the energy difference between the two states $\Delta E = E_i - E_f = hf$.

The Correspondence Principle

a) The correspondence principle states that in situations where the quantisation effects have little effect, classical and quantum calculations should yield the same results. Such an example is for large quantum numbers, n, where the energy levels approach becoming continuous (not discrete), as $E_n = E_0/n^2$.

b) Consider the case of large n and look at the frequency of the photon emitted when an electron makes a transition from the state $n \rightarrow n-1$.

from Bohr's theory,
$$E_n = -\frac{Z^2 E_0}{n^2} \quad \text{where } E_0 = \frac{mk^2e^4}{2\hbar^2}$$
where $n = 1, 2, 3, \ldots$

$$E_{ph} = \left|E_{n-1} - E_n\right| = +Z^2E_0\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right]$$

$$= +Z^2E_0\left[\frac{n^2-(n-1)^2}{n^2(n-1)^2}\right] = +Z^2E_0\left[\frac{n^2-(n^2-2n+1)}{n^2(n-1)^2}\right]$$

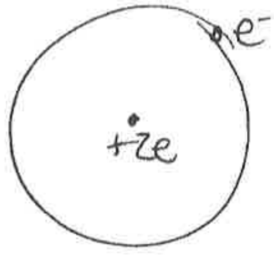
$$= \frac{+Z^2E_0(2n-1)}{n^2(n-1)^2} \quad \text{and } E_{ph} = hf_{qu}.$$

$$\therefore f_{qu} = \frac{Z^2E_0(2n-1)}{n^2(n-1)^2 h} = \frac{Z^2(2n-1)mk^2e^4}{n^2(n-1)^2 2\hbar^2 \cdot h} = \frac{Z^2mk^2e^4(2n-1)}{4\pi\hbar^3 n^2(n-1)^2}$$

c) In the case of large n, $(n-1) \approx n$, $(2n-1) \approx 2n$

$$\therefore f_{qu} \simeq \frac{Z^2mk^2e^4\,2n}{4\pi\hbar^3 n^4} = \frac{Z^2mk^2e^4}{2\pi\hbar^3n^3}$$, as required

d) Consider an electron orbiting the nucleus according to Bohr's theory:



For the n^{th} energy level,

$$L = n\hbar = r_n m v_n \quad \text{(ang. mtm)}$$

also $r_n = \dfrac{n^2 a_0}{Z}$ where $a_0 = \dfrac{\hbar^2}{mke^2}$

now to determine the frequency of oscillation of the electron (also predicted to be the frequency of the radiated photon - classically)

$$f_{cl} = \frac{1}{T}$$ and we know classically,

$$v_n = \frac{\text{distance}}{\text{time}} = \frac{2\pi r_n}{T}$$

$$\therefore f_{cl} = \frac{1}{T} = \left(\frac{2\pi r_n}{v_n}\right)^{-1} = \left(\frac{2\pi r_n}{n\hbar/r_n m}\right)^{-1} = \left(\frac{2\pi m r_n^2}{n\hbar}\right)^{-1}$$

$$= \left[2\pi m \left(\frac{n^2\hbar^2}{Zmke^2}\right)^2 \cdot \frac{1}{n\hbar}\right]^{-1}$$

$$= \frac{n\hbar \cdot Z^2 m^{\cancel{2}} k^2 e^4}{n^4 \hbar^4} \cdot \frac{1}{2\pi \cancel{m}}$$

$$= \frac{Z^2 m k^2 e^4}{2\pi\hbar^3 \cdot n^3} \cong f_{qu}$$ in the case of large quantum number, n.

Thus, both the classical & quantum approach to obtaining the frequency of the emitted photon agree in the large n limit - as required by the correspondence principle.

Tipler & Llewellyn

4-48 To show that a small change in the reduced mass of the electron produces a small change in the spectral line given by

$$\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\mu}{\mu}.$$

Now, the wavelength of the spectral lines is given by

$$\frac{1}{\lambda_{mn}} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$ – the Rydberg-Ritz formula.

Bohr's theory predicts that $R = \frac{mk^2e^4}{4\pi c\hbar^3} \rightarrow \frac{\mu k^2 e^4}{4\pi c\hbar^3}$

where μ = reduced mass.

$\therefore \frac{1}{\lambda} = \mu C$ where $C = \frac{k^2e^4}{4\pi c\hbar^3} \cdot \left(\frac{1}{m^2} - \frac{1}{n^2}\right)$ = constant for a given m & n.

$\therefore \lambda^{-1} = C\mu$ or $\lambda = C'\mu^{-1}$ where C' = constant

differentiating both sides w.r.t μ

$$\therefore \frac{d\lambda}{d\mu} = C' \cdot (-\mu^{-2}) = -\underbrace{(C'\mu^{-1})}_{\lambda}\mu^{-1}$$

$\therefore \frac{d\lambda}{\lambda} = -\frac{d\mu}{\mu}$, as required. ie $\frac{\Delta\lambda}{\lambda} = -\frac{\Delta\mu}{\mu}$

For the case of the Balmer red line, $\lambda = 656.3\,nm$ for hydrogen.

For hydrogen, $\mu_H = \frac{m_e M_H}{m_e + M_H} = \frac{m_e}{1 + m_e/M_H}$

now, $m_e = 9.1094 \times 10^{-31}$ kg

$M_H = 1.6726 \times 10^{-27}$ kg

$M_D = 2M_H$

$\therefore \Delta\lambda = -\frac{\Delta\mu}{\mu} \cdot \lambda$; $\Delta\mu = \left(\frac{m_e M_D}{m_e + M_D}\right) - \left(\frac{m_e M_H}{m_e + M_H}\right)$ D ≡ deuterium

$$= \frac{m_e}{1 + m_e/2M_H} - \frac{m_e}{1 + m_e/M_H} = (0.999728 - 0.999456)\,m_e$$

4-45 (cont^{d})

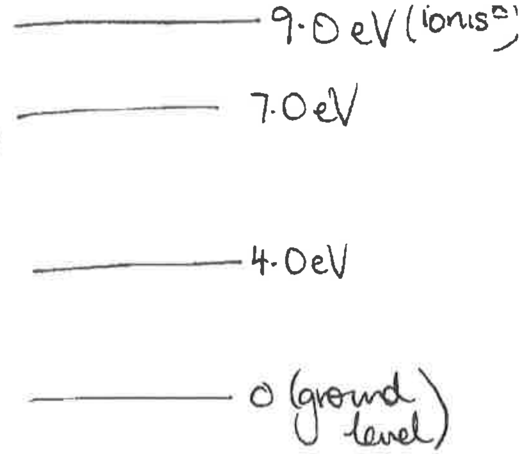
$\therefore \quad \Delta\mu = 0.000272\ m_e$

$\therefore \quad \Delta\lambda = -\dfrac{\Delta\mu}{\mu_H} \cdot \lambda = -\dfrac{0.000272\ m_e}{0.999456\ m_e}\lambda = -0.000272\ \lambda$

656.3 nm

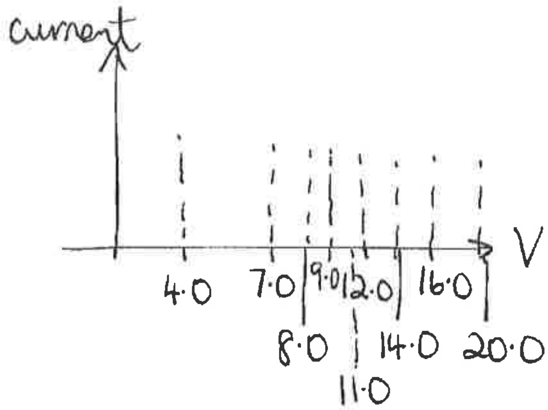
$\Delta\lambda = -0.18\ \text{nm} \quad (-0.179)$

Problem 11



Consider the following hypothetical atom:
It has only 2 excited state (energy levels)

If a vapour of this type of atom was used in the Franck-Hertz experiment,
expect inelastic collisions to occur when can excite electron from ground state to any excited state or ionisation.



at 4.0 ground level electron excites to 4.0 eV level

7.0 " " " 7.0 eV level

8.0 incident electron collides with two atoms, each time losing 4.0 eV of energy (as a consequence of atomic electron being excited to 4.0 eV)

at 9.0 eV atomic electron is ionised
⇒ incident electron loses energy & so current drops.

12.0 incident electron has three collisions, each time losing 4.0 eV to atomic electron.

14.0 incident electron has two collisions, each time losing 7.0 eV to atomic electron

11.0 incident electron has two collisions, once losing 4.0 eV and another losing 7.0 eV to the atomic electron.

etc.

1 collision: 4.0, 7.0

2 collisions: 8.0 ($4+4$), 14.0 ($7+7$), 11.0 ($4+7$), . . .
13.0 ($4+9$), 16.0 ($7+9$), 18.0 ($9+9$)

3 collisions: 12.0 ($4+4+4$), 15.0 ($4+4+7$), 18.0 ($4+7+7$), ~~21.0~~ ($7+7+7$)

4 collisions: 16.0, 19.0, . . .