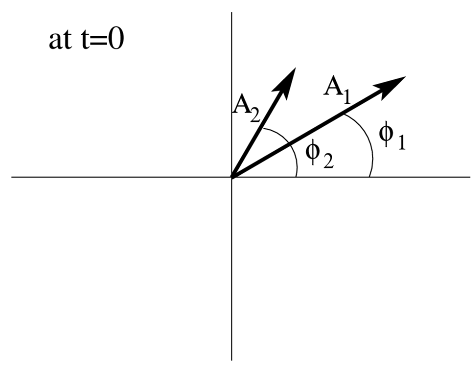
Phasor Handout, Part 2

Adding Phasors

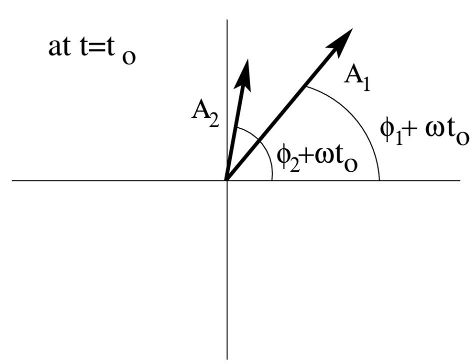
The power and utility of the phasor representation really shows up when combining oscillations. Consider two oscillations, both with the same angular frequency ω, but with different amplitudes and phases:

$$y_1(t) = A_1 \cos(\omega t + \phi_1) \text{ and } y_2(t) = A_2 \cos(\omega t + \phi_2)$$

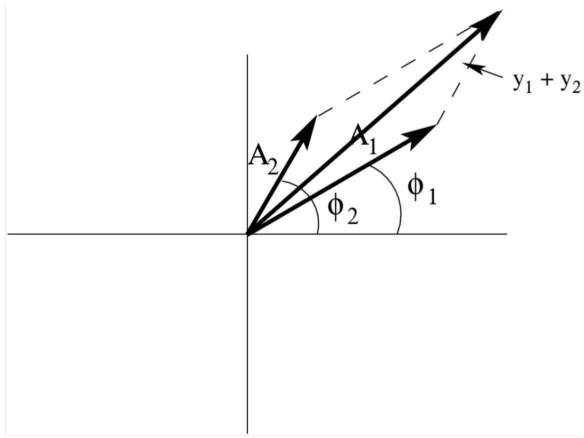
The superposition of these two oscillations $y_{tot} = y_1 + y_2$ is very messy to calculate algebraically; however, it's much simpler using the phasor method. Both oscillations can be placed onto a single phasor diagram,



and since they have the same angular frequency, these phasors will rotate together as time passes. That is, the angle between the phasors will always be $\Delta\phi = \phi_2 - \phi_1$.



The superposition of these two oscillations will also be an oscillation with angular frequency ω. Therefore, we can add the two phasors vectorially and the resultant phasor will describe the superposition of the two oscillations.



The resultant phasor will have its own amplitude and phase, determined from the addition of the two superposed phasors. The amplitude of the resultant will be less than the sum of the two original phasor amplitudes (unless $\Delta\phi = 0$) and the phase of the resultant will be something between the phases of the two original phasors.

Examples

Ex 2-1: You're sitting in a boat in the middle of a calm lake. Suddenly a motor boat drives by, producing waves that would oscillate your boat up and down as follows:

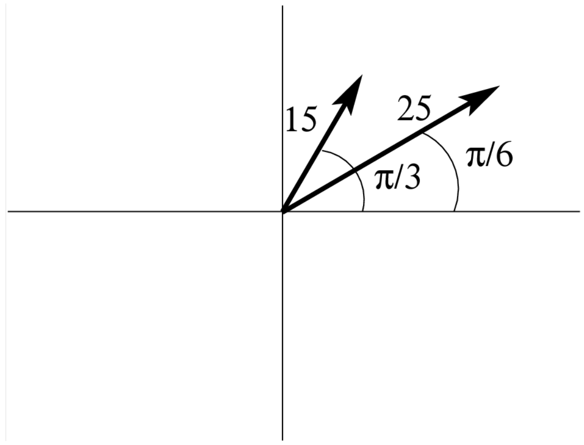
$$y_1(t) = A_1 \cos(\omega t + \phi_1)$$

where $A_1 = 25\,\text{cm}$, $\omega = \frac{2\pi}{3}$, and $\phi_1 = \frac{\pi}{6}$. At the same time, another motor boat drives by, producing waves that would oscillate your boat up and down as follows:

$$y_2(t) = A_2 \cos(\omega t + \phi_2)$$

where $A_2 = 15\,\text{cm}$, $\omega = \frac{2\pi}{3}$, and $\phi_1 = \frac{\pi}{3}$. However, since both waves impact the boat simultaneously, the actual oscillation of your boat is the superposition of these two waves. How much does your boat move up and down as a result of the combination of these waves?

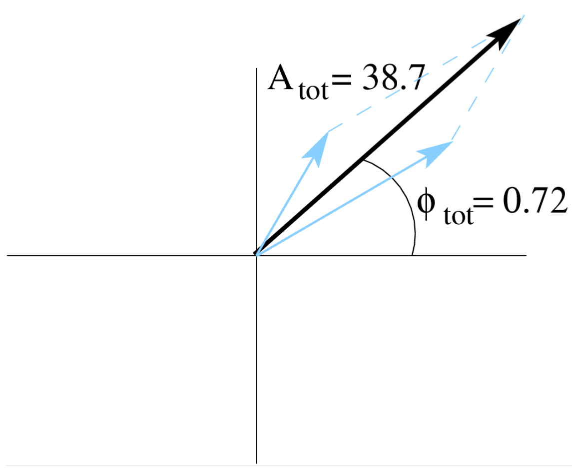
The phasor diagram for these two oscillations looks like this:



The resultant phasor can be determined from the vector addition of the phasors.

Phasor	$\hat{x}$	$\hat{y}$
1	$25\cos\left(\frac{\pi}{6}\right)$ $= 21.6$	$25\sin\left(\frac{\pi}{6}\right)$ $= 12.5$
2	$15\cos\left(\frac{\pi}{3}\right)$ $= 7.5$	$15\sin\left(\frac{\pi}{3}\right)$ $= 13.0$
Total	29.1	25.5

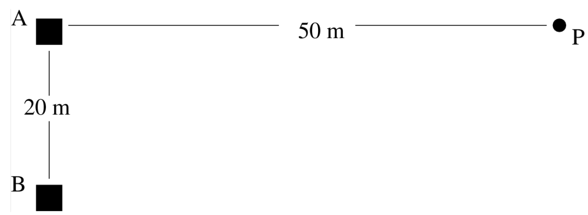
So, the amplitude of the resultant phasor is $A_{tot} = \sqrt{29.1^2 + 25.5^2} = 38.7\,\text{cm}$, and its initial phase is $\phi_{tot} = \tan^{-1}\left(\frac{25.5}{29.1}\right) = 0.72\,\text{rad}$.



We can write the superposition as

$$y_{tot} = 38.7\cos\left(\omega t + 0.72\right).$$

Ex 2-2 Two radio towers, A and B, separated by 20 m, broadcast the same radio signal of wavelength $\lambda = 12\,\text{m}$. You're standing at the point P indicated in the figure with your radio wave amplitude meter.



With only tower A broadcasting, you measure a wave amplitude of 7 (in some unspecified unit). With only tower B broadcasting, you measure a wave amplitude of 5. What amplitude do you measure when both towers are broadcasting?

Here we're interested in the superposition of the waves from the two towers. We aren't explicitly given the phase difference between the towers, but we can figure it out because we know the distances from point P to each tower. If the waves leave the two towers in phase, they won't necessarily be in phase when they reach point P because the waves from tower B have to travel farther. How much farther? Well, the distance between tower B and point P is $\sqrt{50^2 + 20^2} = 53.85\,\text{m}$, so the waves from tower B have to travel 3.85 m farther. That means the waves from tower A will be 3.85 m ahead of the waves from Tower B.

We can turn that distance difference into a phase difference for the waves. The path length difference $\Delta r = 3.85\,\text{m}$ produces $\frac{3.85}{12} = 0.32$ *wavelengths* of difference between the two waves as they arrive at point P.

If the waves from Tower A arrived exactly one wavelength ahead of the waves from Tower B, the signals would add constructively. The phase difference would be 2π, and peaks and troughs of the two waves would be aligned. In this case, however, the waves from Tower A arrive only 0.32 wavelengths ahead of the waves from Tower B. Consequently, the phase difference $\Delta\phi$ is

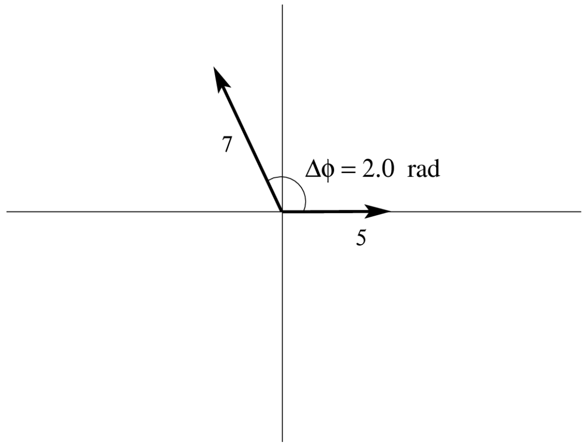
$$\Delta\phi = 2\pi\frac{\Delta r}{\lambda} = 2\pi\frac{3.85}{12} = 2.0\,\text{rad}$$

We can then write expressions for the two oscillations at point P as follows:

$$y_A(t) = 7\cos\left(\omega t + 2.0\right) \qquad y_B(t) = 5\cos\left(\omega t\right)$$

Why did I pick zero initial phase for the signal from Tower B? Because I could. In this case (and in most cases we'll deal with), we're only interested in the phase difference between signals. Therefore, I can always choose to describe one wave with zero initial phase, and then put the phase difference into the expression for the other wave. With one phasor on the horizontal axis, the phasor addition is just easier.

Now we can construct a phasor diagram for the two oscillations,



Once again, the resultant phasor can be determined from the vector addition of the phasors.

Phasor	$\hat{x}$	$\hat{y}$
1	$5\cos(0)$ $= 5$	$5\sin(0)$ $= 0$
2	$7\cos(2.0)$ $= -2.9$	$7\sin(2.0)$ $= 6.4$
Total	2.1	6.4

So, the amplitude of the resultant phasor is $A_{tot} = \sqrt{2.1^2 + 6.4^2} = 6.7$, and its initial phase is $\phi_{tot} = \tan^{-1}\left(\frac{6.4}{2.1}\right) = 1.25\,\text{rad}$.

We can write the superposed signal amplitude as

$$y_{tot} = 6.7\cos(\omega t + 1.25).$$

A 107 - Phasor Addition

Write the superposition of the following two oscillations

$$x_1 = 3\cos\left(\frac{\pi}{3}t\right) \qquad x_2 = 5\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right)$$

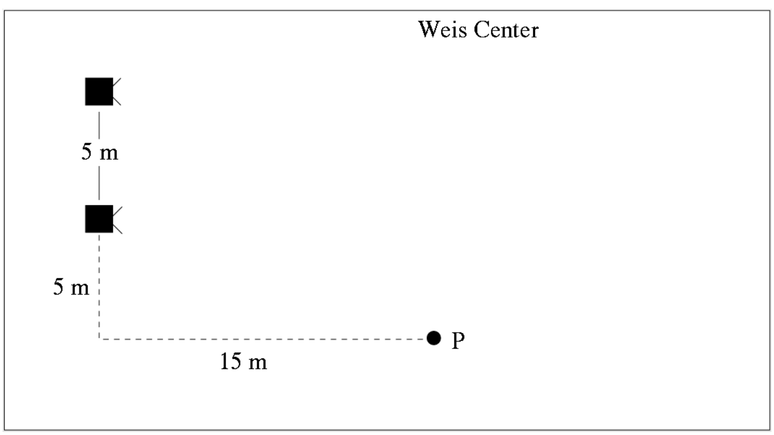
in the form

$$y_3 = A_3\cos\left(\omega t + \phi_3\right)$$

and solve for A_3 and ϕ_3.

A 108 - Sonorous Symphony in C

You're sitting in the Weis Center listening to the "Sonorous Symphony in C," which consists of a single 128 Hz tone played through two speakers separated by 5 m on stage. Both speakers emit sound waves in phase.



You happen to be sitting at the point P in the above diagram, 5 m to the left of the leftmost speaker, and 15 m back from the stage.

If the wave amplitude at your location from each speaker individually is A, what is the amplitude of the combined waves from both speakers at your location?