ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Oct. 9, 2024

1. Bessel functions

   a. Parametric Bessel equation of order ν: $x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0$

   b. Bessel function of the first kind (LI if $\nu \neq$ integer):

$$J_\nu(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{\alpha x}{2}\right)^{2n+\nu} \quad \text{and} \quad J_{-\nu}(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{\alpha x}{2}\right)^{2n-\nu}$$

   c. Bessel function of the second kind (a more general second solution; works even if ν is an integer):

$$Y_\nu(\alpha x) = \frac{\cos(\nu x) J_\nu(\alpha x) - J_{-\nu}(\alpha x)}{\sin(\nu x)}.$$

   Appears to be indeterminate for certain values of x when $\nu =$ integer (evaluates to 0/0). However, it can be shown that Y_ν exists and is LI from J_ν even if ν is an integer (e.g., can use L'Hospital's rule).

   d. A general solution of the parametric Bessel equation:

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

   e. Plots and examples available in "Bessel function" Wikipedia page. See also "Meet the Bessels" by Prof. Maneval for info on Bessel functions of zero order.

2. Inner products of functions and orthogonality – an important ODE/PDE solution tool

   a. Development: inner (dot) product of 3-D vectors and N-dimensional vectors…

$$\mathbf{f} \cdot \mathbf{g} = \mathbf{f}^T \mathbf{g} = f_x g_x + f_y g_y + f_z g_z \qquad \text{and} \qquad \mathbf{f}^T \mathbf{g} = \sum_{j=1}^{N} f_j g_j$$

   inner product of functions over the interval [a, b] (can be thought of as ∞-dimensional vectors)…

$$\mathbf{f}^T \mathbf{g} \quad \rightarrow \quad \lim_{N \to \infty} \frac{b-a}{N} \sum_{j=1}^{N} f(x_j) g(x_j) = \int_a^b f(x) g(x) \, dx$$

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b. One common notation: $\langle f, g \rangle$ denotes an inner product. Context does the rest.

$\langle f, g \rangle = \mathbf{f} \cdot \mathbf{g}$	3-D vectors
$\langle f, g \rangle = \mathbf{f}^T \mathbf{g}$	*N*-dimensional vectors
$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$	functions

c. The next step: If *f* and *g* are functions and $\langle f, g \rangle = 0$, then the functions *f* and *g* are orthogonal. Yes, functions can be orthogonal too.

d. Orthogonality makes possible the *practical* solution of many ODEs and PDEs

e. Unique to inner products of functions – Some inner products require a weight function $w(x)$ to be meaningful. In these cases, the inner product is defined as

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x) \, dx$$

For trigonometric functions (sine and cosine), $w(x) = 1$, but for Bessel functions, $w(x) = x$. We'll see why later.

3. Example: Inner products of trigonometric (circular) functions

a. Eigenfunctions of the Fourier equation (and the wave equation PDE, as we will see soon) over a bounded interval $[-L, L]$ could be

$$y_{1n}(x) = \cos\left(\frac{n\pi x}{L}\right) \quad \text{and/or} \quad y_{2n}(x) = \sin\left(\frac{n\pi x}{L}\right)$$

b. Consider inner product (allow eigenvalues to be different; i.e., use *m* and *n*):

$$\langle y_{1m}, y_{2n} \rangle = \int_{-L}^{L} y_{1m}(x) \, y_{2n}(x) \, dx = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

c. Useful trigonometric identities:

$$\cos(a)\sin(b) = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$$
$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$
$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

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d. Inner product becomes:

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(m+n)\pi x}{L}\right]dx-\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(m-n)\pi x}{L}\right]dx$$

e. Since $\sin(\alpha x)$ is an odd function, if $m \neq n$, then both integrals evaluate to zero. If $m = n$, then

$$\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{2n\pi x}{L}\right]dx-\frac{1}{2}\int_{-L}^{L}\sin\left[\frac{(0)\pi x}{L}\right]dx=0+0=0$$

f. Since $\langle y_{1m}, y_{2n}\rangle = 0$, the functions $y_{1m}(x)$ and $y_{2n}(x)$ are orthogonal for $m \neq n$ and for $m = n$.

g. What about the same eigenfunction with different eigenvalues ($m \neq n$)?

$$\begin{aligned}\langle y_{1m}, y_{1n}\rangle &= \int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx\\ &=\frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(m+n)\pi x}{L}\right]dx+\frac{1}{2}\int_{-L}^{L}\cos\left[\frac{(m-n)\pi x}{L}\right]dx\\ &=0+0 \quad \text{(orthogonal)}\end{aligned}$$

h. …and the same eigenfunction with the same eigenvalues (self-product)?

$$\begin{aligned}\langle y_{1m}, y_{1n}\rangle &= \int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx=\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx\\ &=\int_{-L}^{L}\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2n\pi x}{L}\right)\right]dx=\frac{1}{2}\int_{-L}^{L}dx+\frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2n\pi x}{L}\right)dx\\ &=L+0=L \quad \text{(not orthogonal; self-product is square of "length")}\end{aligned}$$

i. Norm and square norm (general definitions):

$$\|y_n\|=\sqrt{\int_a^b y_n^2(x)\,dx} \quad \text{and} \quad \|y_n\|^2=\int_a^b y_n^2(x)\,dx$$