ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Dec. 9, 2024

1. Policies and review sheet for Final Exam

2. Solution of 2-D Laplace's equation using separation of variables (SoV) method

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   a. Example problem (for a 2-D space that spans $x = 0$ to $x = a$ and $y = 0$ to $y = b$):

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad u(x,0) = 0, \quad \text{and} \quad u(x,b) = f(x)$$

   b. Applying the BCs in x and $u(x, 0) = 0$ yields the general solution

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right).$$

   c. The coefficients $\{A_n\}$ can be determined by applying the second BC in y and exploiting the orthogonality of the eigenfunctions. Multiply the general solution evaluated at $y = b$ by $\cos(m\pi x/a)$ and integrate over the x interval $(0, a)$. The goal is to generate inner products involving the $X_n(x)$ eigenfunctions:

$$u(x,b) = f(x) = A_0 b + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi b}{a}\right)$$

$$\int_0^a f(x)\cos\left(\frac{m\pi x}{a}\right)dx = \int_0^a A_0 b\cos\left(\frac{m\pi x}{a}\right)dx + \int_0^a \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)dx$$

   d. Applying the orthogonality condition for inner products yields

$$A_0 = \frac{1}{ab}\int_0^a f(x)\,dx \quad \text{and} \quad A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

   e. The general solution must satisfy the *maximum principle*, which states that the solution u of Laplace's equation within a bounded region must have its maximum and minimum values on the boundary. There can be no extrema (maxima or minima) within the bounded space.

   f. *Matlab* simulation