ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Nov. 8, 2024

1. SoV solution to wave equation in cylindrical coordinates

$$u(r,t)=\sum_{n=1}^{\infty}\left[A_n\cos(v\alpha_n t)+B_n\sin(v\alpha_n t)\right]J_0(\alpha_n r) \quad \text{with} \quad \sqrt{\lambda_n}=\alpha_n=\frac{x_n}{c}$$

$$A_n=\frac{\langle f(r),J_0(\alpha_n r)\rangle}{\|J_0(\alpha_n r)\|^2} \qquad B_n=\frac{\langle g(r),J_0(\alpha_n r)\rangle}{v\alpha_n\|J_0(\alpha_n r)\|^2}$$

   a. What does the result mean? How are the eigenvalues used and interpreted?

   Vibration frequencies: $\omega_n=2\pi f_n=v\alpha_n=\dfrac{vx_n}{c} \quad\rightarrow\quad f_n=\dfrac{vx_n}{2\pi c}$.
   where x_1 = 2.4048, x_2 = 5.5201, x_3 = 8.6537, x_4 = 11.7915, etc.

   Frequencies are proportional to x_n, but…

   $x_2=2.29x_1,\ x_3=3.59x_1,\ x_4=4.89x_1,\ \ldots$ (no harmonic relationships)

   b. *Matlab* simulation
   c. Are there standing waves? How do they compare to the vibrating string case?

2. Next: Numerical solution of PDEs using finite differences

   a. First derivative approximations:

      i. Forward difference: $\dfrac{df(x_o)}{dx}\approx\dfrac{f(x_o+\Delta x)-f(x_o)}{\Delta x}$

      ii. Backward difference: $\dfrac{df(x_o)}{dx}\approx\dfrac{f(x_o)-f(x_o-\Delta x)}{\Delta x}$

      iii. Centered difference: $\dfrac{df(x_o)}{dx}\approx\dfrac{f(x_o+0.5\Delta x)-f(x_o-0.5\Delta x)}{\Delta x}$

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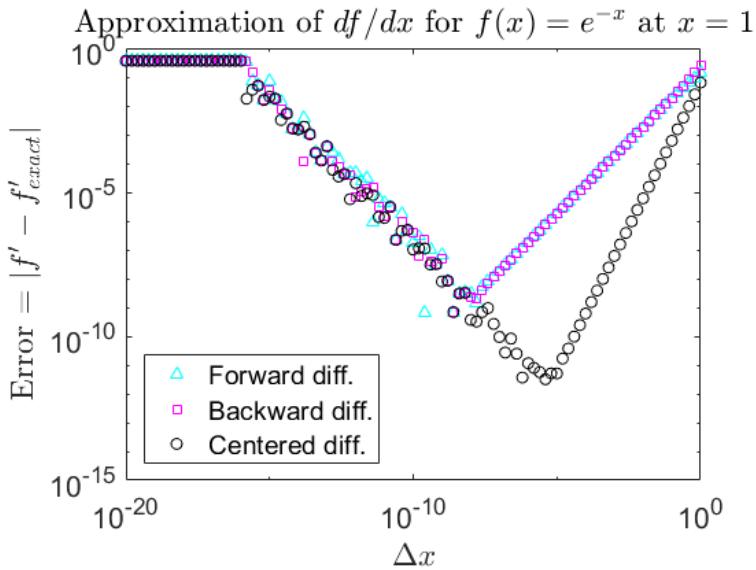
iv. Error comparison (Δx larger to the right in graph):



b. Second derivative approximation: $\dfrac{d^2 f(x_o)}{dx^2} \approx \dfrac{f(x_o+\Delta x) - 2f(x_o) + f(x_o - \Delta x)}{\Delta x^2}$

c. Example: For $f(x) = e^x$, approximate $f'(1.2)$ using finite differences with Δx = 0.1, 0.05, and 0.01.

d. Example: For $f(x) = e^x$, approximate $f'(1.2)$ using centered finite differences with Δx = 0.1, 0.05, and 0.01. Exact result (to 5 sig. digs.) is $f'(1.2)$ = 3.3201.

i. Δx = 0.1: $f'(1.2) \approx \dfrac{f(1.2+0.05) - f(1.2-0.05)}{0.1} = \dfrac{e^{1.25} - e^{1.15}}{0.1} = 3.3215$

ii. Δx = 0.05: $f'(1.2) \approx \dfrac{f(1.2+0.025) - f(1.2-0.025)}{0.05} = \dfrac{e^{1.225} - e^{1.175}}{0.05} = 3.3205$

iii. Δx = 0.01: $f'(1.2) \approx \dfrac{f(1.2+0.005) - f(1.2-0.005)}{0.01} = \dfrac{e^{1.205} - e^{1.195}}{0.01} = 3.3201$

3. Application: Finite difference solution of the heat equation

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{where } c = \text{thermal diffusivity}$$

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a. Questions:

   i. What do finite difference approximations look like when there is more than one independent variable?
   ii. How many solution points (in x and in t) do we select?
   iii. What do we do about the boundaries?

b. Finite difference approximations of partial derivatives (hold nondifferentiated variable constant)

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \quad \text{and} \quad \frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

c. Note that the x-derivative is approximated using a centered difference and the time derivative by a forward difference. We will see why very soon.

d. Heat equation expressed using finite differences

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$