ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Oct. 7, 2024

1. Fourier equation as an eigenvalue problem; three possible solution forms (for closed boundaries)

$$y'' + \lambda y = 0$$

   a. If $\lambda < 0$: $y(x) = c_1 \cosh\left(\sqrt{-\lambda}x\right) + c_2 \sinh\left(\sqrt{-\lambda}x\right)$
   b. If $\lambda = 0$: $y(x) = c_1 + c_2 x$
   c. If $\lambda > 0$: $y(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$

2. Other ODE classes with variable coefficients (2nd order only)

   a. Parametric Bessel equation of order ν (parameter is α)

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0$$

   b. Modified parametric Bessel equation of order ν

$$x^2 y'' + xy' - \left(\alpha^2 x^2 + \nu^2\right) y = 0$$

   c. Legendre equation of order n

$$\left(1 - x^2\right) y'' - 2xy' + n\left(n+1\right) y = 0$$

   d. Airy equation

$$y'' \pm a^2 xy = 0$$

   e. Chebyshev equation

$$\left(1 - x^2\right) y'' - xy' + a^2 y = 0$$

   f. Need more sophisticated solution methods than those used for ODEs with constant coefficients.

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3. A general approach for solving ODEs with variable coefficients:

   a. Power series solution

   $$y(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^n ,$$

   but most of the ones we will see are expanded about $x_o = 0$:

   $$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

   b. Power series solutions valid only over intervals for which they converge and are analytic; they are often not valid over all of $-\infty < x < \infty$
   c. Example: $\ln(x)$ is not analytic for $x \le 0$, so the ordinary points are $x > 0$
   d. Ordinary points: Values of x at which variable coefficients are analytic
   e. Singular points: Values of x that are not ordinary points. Test (for the 2nd-order case):

   ODE with variable coefficients: $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

   Standard form: $y'' + P(x)y' + Q(x)y = 0$,

   where $P(x) = \dfrac{a_1(x)}{a_2(x)}$ and $Q(x) = \dfrac{a_0(x)}{a_2(x)}$.

   f. Ordinary point at $x = x_0$ if $a_2(x_0) \neq 0$; otherwise, it's a singular point
   g. Singular points can regular or irregular
      i. Regular if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$
      ii. Irregular if not
   h. Obtaining power series solutions about singular points requires special approaches. For regular singular points, the method of Frobenius (Sec. 5.2) is one possibility.

4. Example: Parametric Bessel equation of order ν

   $$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

   a. The "parameter" is α
   b. Series solution (just one of the two possible solutions) is the *Bessel function of the first kind*,

   $$J_\nu(\alpha x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{\alpha x}{2}\right)^{2n+\nu} ,$$

   where $\Gamma(1 + \nu)$ is the gamma function (more info in Appendix II of Zill, 6th ed.)

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c. Second series solution, which is LI from J_ν if $\nu \neq$ integer:

$$J_{-\nu}\left(\alpha x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{n!\Gamma\left(1-\nu+n\right)}\left(\frac{\alpha x}{2}\right)^{2n-\nu}$$

d. There are many practical applications with $\nu =$ integer. A more general second solution is the *Bessel function of the second kind*,

$$Y_\nu\left(\alpha x\right)=\frac{\cos\left(\nu x\right)J_\nu\left(\alpha x\right)-J_{-\nu}\left(\alpha x\right)}{\sin\left(\nu x\right)}.$$

Appears to be indeterminate for certain values of x when $\nu =$ integer (get 0/0). However, it can be shown that Y_ν exists and is LI from J_ν even if ν is an integer (e.g., can use L'Hospital's rule).

e. Thus, the general solution of the parametric Bessel equation can be written

$$y\left(x\right)=c_1J_\nu\left(\alpha x\right)+c_2Y_\nu\left(\alpha x\right)$$

f. The J_ν and Y_ν functions are well known and tabulated in old references (see web link: NIST Digital Library of Mathematical Functions), but most modern math software packages have Bessel functions built in (e.g., in *Matlab*, they are `besselj` and `bessely`).

g. See "Meet the Bessels" by Prof. Maneval for info on Bessel functions of zero order.