ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Sept. 6

1. Solvability of M-by-N systems (see flowchart in Fig. 8.3.1)

   a. M = no. of equations; N = no. of unknowns
   b. $r = \text{rank}(A) = \text{rank}(A|\mathbf{b})$: consistent & solution possible
      i. $r = N$: unique solution
      ii. $r < N$: infinitely many solutions
   c. $r = \text{rank}(A) < \text{rank}(A|\mathbf{b})$: inconsistent & no solution possible
   d. One more thing: $\text{rank}(A) = \text{rank}(A^T)$

2. Triangulation example:

   a. Boat at location (x_o, y_o) – two unknowns x_o and y_o
   b. Direction finders (sensors) at locations (x_1, y_1), (x_2, y_2), and (x_3, y_3) – coordinates are known
   c. ϕ = angle to boat relative to x-axis
   d. System of equations (A is 3×2; $\mathbf{b}$ is 3×1):

$$\begin{bmatrix} \sin\phi_1 & -\cos\phi_1 \\ \sin\phi_2 & -\cos\phi_2 \\ \sin\phi_3 & -\cos\phi_3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \sin\phi_1 - y_1 \cos\phi_1 \\ x_2 \sin\phi_2 - y_2 \cos\phi_2 \\ x_3 \sin\phi_3 - y_3 \cos\phi_3 \end{bmatrix}$$

   e. Examples using *Matlab* script `TriangulationSolvabilityDemo.m`.
   f. System is overdetermined ($M = 3$ sensors, $N = 2$ boat coordinates).
   g. Could have $\text{rank}(A) = \text{rank}(A|\mathbf{b}) = 1 \rightarrow$ infinitely many solutions. (When?)
   h. If $\text{rank}(A) = N = 2$, could have $\text{rank}(A|\mathbf{b}) = 2$ (one solution; consistent) or 3 (no solutions; inconsistent). (Why?)

3. Generality #1: Overdetermined systems are usually (but not always) inconsistent.

Examples: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

Although an overdetermined system might not have an exact solution, it could still have a "best" approximate solution.

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4. Generality #2: Underdetermined systems are usually (but not always) consistent:

   Examples: $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$ $\qquad$ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

   Underdetermined systems can have infinitely many solutions or no solution but never a unique solution because rank(A) $\leq M < N$ always.

5. New topic: Curve-fitting and the method of least squares

   a. Start with an example. Consider the following small data set. How can we estimate the value of $y(3)$, that is, the value of y at $x = 3$?

i	x_i	y_i
1	1.0	1.1
2	2.0	3.2
3	4.0	5.2

   b. One possible approach: Set up a matrix expression that computes the coefficients of the quadratic expression for a curve that passes through the data points.

   $$y = c_0 + c_1 x + c_2 x^2$$

   Is the matrix equation solvable?
   If so, is the solution acceptable?

   c. Another possible approach: Set up a matrix expression that computes the coefficients of the linear expression for a line that passes through the data points.

   $$y = c_0 + c_1 x$$

   Is the matrix equation solvable?
   If so, is the solution acceptable?

   d. Next: a general approach applicable to any set of functions or curves
      i. Is a quadratic fit appropriate? Does it correspond to the nature of the problem or data set?
      ii. Is a linear fit appropriate?
      iii. The elementary functions should match the "physics" of the problem and how the solution will be used. Should the fit to the curve be a staircase function, a piecewise linear function, or smoothly varying? If smooth, what types of elementary functions are appropriate? Sinusoids? Exponentials? Gaussian? Something else?