ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Nov. 6, 2024

1. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius *c*, assuming only radial vibrational modes (no angle-dependent modes):

$$v^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

   a. ODEs after separation (R equation is a parametric Bessel equation of order 0)

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad \text{and} \quad T'' + \lambda v^2 T = 0$$

   b. Boundary conditions and initial conditions and their interpretation

$$u(c,t)=0\,, \quad u(r,0)=f(r)\,, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r)\,; \quad u \text{ is finite everywhere}$$

   a. Suggested solution forms for ODEs (with $\lambda_n = \alpha_n^2$)

$$R_n(r)=c_1 J_0(\alpha_n r)+c_2 Y_0(\alpha_n r) \quad \text{and} \quad T_n(t)=c_3\cos(v\alpha_n t)+c_4\sin(v\alpha_n t)$$

   b. The $R(r)$ problem is an eigenvalue problem (it's also a BVP) and specifically a singular Sturm-Liouville problem; the $T(t)$ problem is an IVP
   c. Special additional condition: Solution must be finite within boundary (for $r \le c$)
   d. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.

2. SoV solution and its interpretation (x_n = roots of J_0)

$$u(r,t)=\sum_{n=1}^{\infty}\left[A_n\cos(v\alpha_n t)+B_n\sin(v\alpha_n t)\right]J_0(\alpha_n r) \quad \text{with} \quad \sqrt{\lambda_n}=\alpha_n=\frac{x_n}{c}$$

   a. Inner product of Bessel functions; weighting function is $p(r) = r$

$$\left\langle J_0(\alpha_m r), J_0(\alpha_n r)\right\rangle = \int_0^c r\, J_0(\alpha_m r) J_0(\alpha_n r)\, dr$$

(*continued on next page*)

b. Apply ICs to find expressions for $\{A_n\}$ and $\{B_n\}$ coefficients

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} \left[A_n(1) + B_n(0) \right] J_0(\alpha_n r)$$

$$\rightarrow \int_0^c r\, f(r) J_0(\alpha_m r)\, dr = \sum_{n=1}^{\infty} A_n \int_o^c r\, J_0(\alpha_n r) J_0(\alpha_m r)\, dr \quad \rightarrow \quad A_n = \frac{\langle f(r), J_0(\alpha_n r) \rangle}{\left\| J_0(\alpha_n r) \right\|^2}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[-A_n v\alpha_n \sin(v\alpha_n t) + B_n v\alpha_n \cos(v\alpha_n t) \right] J_0(\alpha_n r)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r) \quad \rightarrow \quad B_n = \frac{\langle g(r), J_0(\alpha_n r) \rangle}{v\alpha_n \left\| J_0(\alpha_n r) \right\|^2}$$

c. What does the result mean? How are the eigenvalues used and interpreted?

Vibration frequencies: $\omega_n = 2\pi f_n = v\alpha_n = \frac{v x_n}{c} \quad \rightarrow \quad f_n = \frac{v x_n}{2\pi c}$.
where $x_1 = 2.4048$, $x_2 = 5.5201$, $x_3 = 8.6537$, $x_4 = 11.7915$, etc.

Frequencies are proportional to x_n, but…

$x_2 = 2.29 x_1$, $x_3 = 3.59 x_1$, $x_4 = 4.89 x_1$, … (no harmonic relationships)

d. *Matlab* simulation
e. Are there standing waves? How do they compare to the vibrating string case?