ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Dec. 6, 2024

1. Solution of Laplace's equation using separation of variables (SoV) method

   a. Laplacian operator:

   $$\text{3-D: } \nabla^2 u = 0 \quad \rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

   $$\text{2-D: } \nabla^2 u = 0 \quad \rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   b. These are typically steady-state problems (often involving heat or static electric charge); there is no time variation. The quantity *u* (like heat or charge) has distributed itself and achieved a self-sustaining state.

   c. General SoV approach (2-D case):

   $$\text{Let } u(x,t) = X(x)Y(y)$$

   Substitution leads to

   $$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0 \quad \rightarrow \quad X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

   Two linked ODEs:

   $$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad -\frac{Y''}{Y} = -\lambda \quad \rightarrow \quad Y'' - \lambda Y = 0$$

   Note that in the basic 2-D case, one ODE will be a Fourier equation and the other will be a modified Fourier equation.

   d. Generally, there are two boundary conditions for each spatial coordinate in any combination of types (Dirichlet, Neumann, or Robin). As an example, consider the following combination for a 2-D space that spans $x = 0$ to $x = a$ and $y = 0$ to $y = b$:

   $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad u(x,0) = 0, \quad \text{and} \quad u(x,b) = f(x)$$

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e. Since the X problem involves a bounded space, the recommended solution form for $\lambda > 0$ (only the trivial solution exists for $\lambda < 0$) is

$$X(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right),$$

and its derivative is

$$X'(x) = -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right).$$

Applying the BCs in x yields

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \rightarrow \quad X'(0)Y(y) = 0 \quad \rightarrow \quad X'(0) = 0 = -c_1\sqrt{\lambda}(0) + c_2\sqrt{\lambda}(1) \quad \rightarrow \quad c_2 = 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \quad \rightarrow \quad X'(a)Y(y) = 0 \quad \rightarrow \quad X'(a) = 0 = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda} a\right) \quad \rightarrow \quad \sqrt{\lambda} a = n\pi,$$

for n = 1, 2, 3, … in the case of the second BC. If $\lambda = 0$, then

$$X_0(x) = c_1 + c_2 x \quad \text{and} \quad X_0'(x) = c_2 .$$

Applying the BCs in x yields

$$X_0'(0) = 0 = c_2 \quad \rightarrow \quad c_2 = 0$$
$$X_0'(a) = 0 = c_2 \quad \rightarrow \quad c_2 = 0 \text{ (again)},$$

but coefficient c_1 is still indeterminate and could be nonzero. That is, if $\lambda = 0$, then its corresponding eigenfunction $X_0(x)$ could be a nonzero constant. We can combine the $\lambda = 0$ and $\lambda > 0$ cases to obtain the eigenfunctions

$$X_n(x) = c_{1n} \cos\left(\frac{n\pi x}{a}\right), \quad \text{for } n = 0, 1, 2, 3, \ldots$$

f. The Y problem involves a modified Fourier equation over a bounded space, so the recommended solution form for $\lambda_n > 0$ (we already know that $\lambda_n = (n\pi/a)^2$) is

$$Y_n(y) = c_{3n} \cosh\left(\sqrt{\lambda_n} y\right) + c_{4n} \sinh\left(\sqrt{\lambda_n} y\right),$$

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Applying the first BC in y yields

$$u(x,0)=0 \quad \rightarrow \quad X_n(x)Y_n(0)=0 \quad \rightarrow \quad Y_n(0)=0=c_{3n}(1)+c_{4n}(0) \quad \rightarrow \quad c_{3n}=0,$$

so

$$Y_n(y)=c_{4n}\sinh\left(\frac{n\pi y}{a}\right), \quad \text{for } n = 0, 1, 2, 3, \ldots$$

If $\lambda = 0$, then

$$Y_0(y)=c_5+c_6 y.$$

Applying the first BC in y yields

$$Y_0(0)=0=c_5+c_6(0) \quad \rightarrow \quad c_5=0$$

but coefficient c_6 is still indeterminate, so $Y_0(y) = c_6 y$.

g. The general solution is

$$u(x,y)=A_0 y+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right),$$

where coefficients have been combined as $A_0 = c_{1,0}\, c_6$ and $A_n = c_{1n}\, c_{4n}$.

h. At this point, the coefficients $\{A_n\}$ are unknown, but they can be determined by applying the second BC in y and then exploiting the orthogonality of the eigenfunctions. Multiplying the general solution evaluated at $y = b$ by $\cos(m\pi x/a)$ and integrating over the x interval $[0, a]$ yields

$$\int_0^a f(x)\cos\left(\frac{m\pi x}{a}\right)dx=\int_0^a A_0 b\cos\left(\frac{m\pi x}{a}\right)dx+\int_0^a \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)dx,$$

which produces the expressions for the coefficients given by

$$A_0=\frac{1}{ab}\int_0^a f(x)\,dx \quad \text{and} \quad A_n=\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx.$$

i. The general solution must satisfy the *maximum principle*, which states that the solution u of Laplace's equation within a bounded region must have its maximum and minimum values on the boundary. There can be no extrema (maxima or minima) within the bounded space.

j. *Matlab* simulation