ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Sept. 4

1. Singular systems: rank and consistency

   a. Consistent: has at least one solution or infinitely many solutions (other numbers of solutions – e.g., 3 – are not possible). Example in 2-D: Solution space is a point or line
   b. Inconsistent: No solutions: Example in 2-D: Parallel lines – no intersecting point
   c. Rank: Max. no. of independent row vectors in a matrix (can use `rank` command in *Matlab*) – examples coming soon
   d. Relationship between rank and row-echelon (or reduced row-echelon) form

2. Examples of consistent and inconsistent solutions

$$\begin{aligned} x+2y-5z&=2 \\ 2x-3y+4z&=4 \\ 4x+y-6z&=8 \end{aligned} \quad\rightarrow\quad \begin{bmatrix} 1 & 2 & -5 \\ 0 & -7 & 14 \\ 0 & -7 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 2 & -5 \\ 0 & -7 & 14 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

   All zeros on third line above implies that there are infinitely many solutions (consistent).

$$\begin{aligned} x+2y-5z&=1 \\ 2x-3y+4z&=2 \\ 4x+y-6z&=3 \end{aligned} \quad\rightarrow\quad \begin{bmatrix} 1 & 2 & -5 \\ 0 & -7 & 14 \\ 0 & -7 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & 2 & -5 \\ 0 & -7 & 14 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

   Nonzero entry on right-hand side of third line above implies that there are no solutions (inconsistent).

3. Rank of a matrix

   a. For square matrices, rank(A) = max. no. of linearly independent row vectors in A
   b. For $M \times N$ matrices, it turns out that rank(A) = the max. no. of linearly independent row *or* column vectors in A
   c. intimately related to the solvability and consistency of linear systems

4. Views of the solution to $A\mathbf{x} = \mathbf{b}$

   a. Equation view: Each row is a "line" in R^n. Do they intersect?
   b. Vector view: Combination of the columns of A. Can they synthesize $\mathbf{b}$?
   c. rank(A): No. of independent equations vs. no. of independent row vectors
   d. rank($A|\mathbf{b}$): System consistency. Does $\mathbf{b}$ "add" anything?

(*continued on next page*)

5. Solvability of M-by-N systems (handout: "Solvability Flowchart for Rectangular Systems")

   a. M = no. of equations; N = no. of unknowns; r = rank(A)
   b. If r = rank($A|\mathbf{b}$): consistent & solution possible
      i. $r = N$: unique solution
      ii. $r < N$: infinitely many solutions
   c. If r < rank($A|\mathbf{b}$): inconsistent & no solution possible

<u>Example Problems Involving Rank and Solvability</u>

2 × 2 systems:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

3 × 2 systems:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Although an overdetermined system might not have an exact solution, it could still have a "best" approximate solution.

2 × 3 systems:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Underdetermined systems can have infinitely many solutions or no solution but never a unique solution because rank(A) $\le M < N$ always.