ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Oct. 4, 2024

1. Boundary value problems (BVPs) involving special DEs

   a. Primarily concerned with 2nd order ODEs (most common in mathematical physics)
   b. Appear frequently in important partial differential equations (PDEs):
      i. Fourier equation and modified Fourier equation
      ii. Cauchy-Euler equation
      iii. Bessel equation (including parametric and modified forms)
   c. Others (Legendre, Airy, …) appear less frequently but have important special applications

2. Solutions to Fourier and modified Fourier equations:

$$y'' + a^2 y = 0 \qquad \text{and} \qquad y'' - a^2 y = 0$$

   a. For closed boundaries (i.e., problem defined over finite range of x), recommend

$$y(x) = c_1 \cos(ax) + c_2 \sin(ax) \qquad \text{and} \qquad y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   Roots r_1 and r_2 of characteristic equation imaginary for Fourier equation and real for modified Fourier equation

   b. For open boundaries (i.e., problem defined over infinite or semi-infinite range of x), recommend

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

3. Example #1: BVP involving Fourier equation:

$$y'' + a^2 y = 0 \quad \text{with} \quad y(0) = 0,\ y(1) = 0$$

   a. Nontrivial solution is $y(x) = c_2 \sin(n\pi x)$, $n = 1, 2, 3, \ldots$
   b. Infinitely many nontrivial solutions since infinitely many integers n will work. This is an eigenvalue problem. Constants $a_n = n\pi$ are *eigenvalues*, and elementary solutions $\sin(n\pi x)$ are *eigenfunctions*.
   c. Compare $y'' = -a^2 y$ to $A\mathbf{y} = \lambda\mathbf{y}$, where A is a linear operator. (A 2nd order derivative is also a linear operator.)
   d. The constant c_2 is left unspecified in this problem. However, if there had been a forcing function [i.e., $y'' + a^2 y = g(x)$], then c_2 could be uniquely specified.

(*continued on next page*)

4. Example #2: Now consider an arbitrary value for a^2 but the same BCs:

$$y'' + 2.5\pi y = 0 \quad \text{with} \quad y(0) = 0\,,\ y(1) = 0$$

No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.

5. Example #3: BVP involving modified Fourier equation:

$$y'' - a^2 y = 0 \quad \text{with} \quad y(0) = 0\,,\ y(1) = 0$$

   a. Attempt to apply

$$y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$$

   b. No nontrivial solutions because $y(1) = 0$ is not satisfied; $y = 0$ is still a solution.

6. Cauchy-Euler equation

$$ax^2 y'' + bxy' + cy = 0$$
$$\text{special case: } x^2 y'' + xy' - \alpha^2 y = 0$$
$$\text{special special case: } x^2 y'' + xy' - y = 0$$

7. "Peel-the-onion" method applied to Cauchy-Euler equation with $a = b = 1$ and $c = -1$

$$x^2 y'' + xy' - y = 0 \quad \text{equiv. to} \quad \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}(xy)\right] = 0$$

Successive integrations to arrive at solution. First integration w.r.t. x:

$$\frac{1}{x}\frac{d}{dx}(xy) = c_1 \quad \rightarrow \quad \frac{d}{dx}(xy) = c_1 x$$

Second integration w.r.t. x:

$$xy = c_1\frac{x^2}{2} + c_2 \quad \rightarrow \quad y = c_1\frac{x}{2} + c_2\frac{1}{x}$$

8. Solution to 2nd order Cauchy-Euler equation with $a = b = 1$ (See Sec. 3.6 of Zill, 6th ed.; see also Sec. 3.2, which explains the reduction of order method for finding a second independent solution)

$$x^2 y'' + xy' - \alpha^2 y = 0 \quad \text{solutions are} \quad y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$$