ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Nov. 4, 2024

1. Interpretation of wave equation solution

   a. *Matlab* simulation
   b. Vibration modes and resonances; harmonically related
   c. Standing waves vs. traveling waves. For example, consider the $g(x) = 0$ (or $B_n = 0$) case. Each eigenmode, which is a *standing* wave, can be expressed as the sum of two counterpropagating *traveling* waves with speed v:

   $$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) = \frac{1}{2}\sin\left[\frac{n\pi}{L}(x+vt)\right] + \frac{1}{2}\sin\left[\frac{n\pi}{L}(x-vt)\right]$$

   All eigenmodes propagate at the same speed a in this simple example. In more realistic problems, different eigenmodes travel at different speeds (dispersion).

   d. Not modeled in this example:

      i. Acoustic coupling between strings and nearby non-fixed objects
      ii. Energy dissipation within string and nearby objects and due to air resistance
      iii. Scattering (reflections) from nearby objects
      iv. Presence of nearby objects, especially resonant cavities, which can affect sound perceived by listeners and can alter resonant frequencies somewhat
      v. Gravity

2. Wave equation (1-D) problems with open boundaries (or boundaries so far away that they can be considered to be infinite in extent)

   $$v^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad -\infty \le x \le \infty, \quad t \ge 0$$

   $$\text{ICs:} \quad u(x,0) = f(x) \quad \text{and} \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$$

   a. Special solution methods required (e.g., Green’s functions for nonhomogeneous problems)
   b. One approach: See supplemental reading on D’Alembert’s solution
   c. Solution has $x \pm vt$ in arguments of functions (traveling waves)

   $$u(x,t) = \frac{1}{2}f(x+at) + \frac{1}{2}f(x-at) + \frac{1}{2a}g_A(x+at) - \frac{1}{2a}g_A(x-at),$$

   where $g_A(x)$ is the antiderivative (integral) of $g(x)$.

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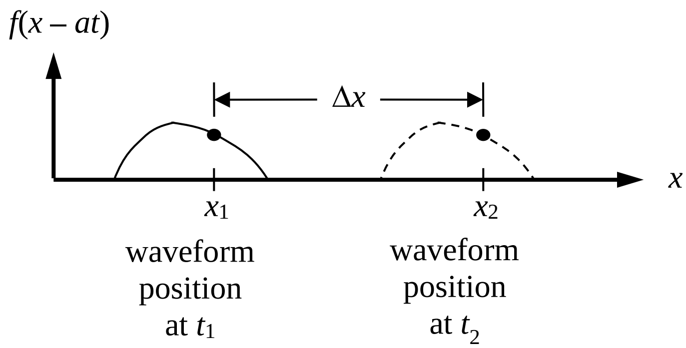
d. Example: depiction of propagation of the waveform $f(x - at)$ over time:



The point on the waveform marked with a dot is at $x = x_1$ at time t_1. At later time t_2, the waveform advances to the right so that the dot reaches the location $x = x_2$. The waveform travels the distance $\Delta x = x_2 - x_1$ over the time interval $\Delta t = t_2 - t_1$. The dot represents a particular point along the function $f(x)$, so

$$x_2 - at_2 = x_1 - at_1 \quad \rightarrow \quad x_2 - x_1 = a(t_2 - t_1) \quad \rightarrow \quad \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = a \qquad (a = \text{speed})$$

3. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius c, assuming only radial vibrational modes (no angle-dependent modes):

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

a. ODEs after separation (R equation is a parametric Bessel equation of order 0)

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad \text{and} \quad T'' + \lambda a^2 T = 0$$

b. Boundary conditions and initial conditions and their interpretation

$$u(c,t) = 0, \quad u(r,0) = f(r), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r); \quad u \text{ is finite everywhere}$$

a. Suggested solution forms for ODEs (with $\lambda_n = \alpha_n^2$)

$$R_n(r) = c_1 J_0(\alpha_n r) + c_2 Y_0(\alpha_n r) \quad \text{and} \quad T_n(t) = c_3\cos(a\alpha_n t) + c_4\sin(a\alpha_n t)$$

b. The $R(r)$ problem is an eigenvalue problem (it's also a BVP) and specifically a singular Sturm-Liouville problem; the $T(t)$ problem is an IVP
c. Special additional condition: Solution must be finite within boundary (for $r \le c$)
d. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.