ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Sept. 30, 2024

1. Singular value decomposition (SVD): Implications and properties

   a. $A = U\Sigma V^T = \sum_{j=1}^{N} \sigma_j \mathbf{u}_j \mathbf{v}_j^T$, where $\mathbf{u}_j \mathbf{v}_j^T$ is an outer product, each of which is $M \times N$.
      i. The outer products have rank = 1. (See HW #2 Probs. 4 & 5.)
      ii. A is a weighted sum of rank-1 $M \times N$ matrices.
      iii. The weights (σ_j) grow progressively smaller as j increases.
      iv. If any of the singular values are zero or too small to matter (lost in the noise, for example), then A can be represented by two compact sets of orthogonal vectors $\{\mathbf{u}_i\}_{i=1 \text{ to } r}$ and $\{\mathbf{v}_i\}_{i=1 \text{ to } r}$, where $r < N$.
      v. The summation terms for zero or nearly zero singular values can be ignored. (Example given in image processing demo later.)

   b. Use complex conjugate transpose (Hermitian) for complex matrix A.

   c. U and V are both orthogonal ($U^{-1} = U^T$ and $V^{-1} = V^T$); thus,
   $A = U\Sigma V^T \quad \rightarrow \quad AV = U\Sigma \quad \rightarrow \quad A\mathbf{v}_i = \sigma_i \mathbf{u}_i$

   d. $A = U\Sigma V^T \quad \rightarrow \quad A^T = V\Sigma^T U^T \quad \rightarrow \quad A^T U = V\Sigma \quad \rightarrow \quad A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$

   e. $A^T A = \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$
   $\rightarrow \quad \left(A^T A\right)V = V\Sigma^T\Sigma \quad \rightarrow \quad \left(A^T A\right)\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$
   σ_i^2 are the nonzero eigenvalues of $A^T A$, and $\{\mathbf{v}_i\}_{i=1 \text{ to } N}$ are the eigenvectors

   f. $AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$
   $\rightarrow \quad \left(AA^T\right)U = U\Sigma\Sigma^T \quad \rightarrow \quad \left(AA^T\right)\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$
   σ_i^2 are also the nonzero eigenvalues of AA^T, and $\{\mathbf{u}_i\}_{i=1 \text{ to } M}$ are the eigenvectors

   g. If A is symmetric, then $A^T A = AA^T = A^2$, so $\lambda_i^2 = \sigma_i^2 \quad \rightarrow \quad |\lambda_i| = |\sigma_i|$ (sign ambiguity)

2. Applications and examples

   a. To solve a system $A\mathbf{x} = \mathbf{b}$ (this is for overdetermined and square systems; underdetermined requires interpretation):

   $$U\Sigma V^T \mathbf{x} = \mathbf{b} \quad \rightarrow \quad \Sigma V^T \mathbf{x} = U^T \mathbf{b} \quad \rightarrow \quad V^T \mathbf{x} = \Sigma^{-1} U^T \mathbf{b} \quad \rightarrow \quad \mathbf{x} = V\Sigma^{-1} U^T \mathbf{b}$$

(*continued on next page*)

Since Σ is diagonal (in the "economy-sized" decomposition), its inverse is

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_N \end{bmatrix}_{N\times N}.$$

Equivalent to

$$\mathbf{x} = \sum_{j=1}^{N} \left(\frac{\mathbf{u}_j^T \mathbf{b}}{\sigma_j} \right) \mathbf{v}_j$$

In ill-conditioned systems, some **u** vectors are nearly orthogonal to **b**. (In singular systems, some *are* orthogonal.) The corresponding singular values are very small, so those terms "corrupt" the solution **x**. Workaround: Omit the problematic terms with small σ_j (see Lab #4).

b. Example #1: Compare eigenvalues to singular values of the symmetric matrix A (use *Matlab* `eig` and `svd` commands):

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

i. Notice order of singular values
ii. Check orthogonality of columns of U and V
iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
iv. What is the rank of this matrix?
v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

c. Example #2: Compare eigenvalues to singular values of the singular matrix A:

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ 4 & 1 & -6 \end{bmatrix}$$

i. Notice order of singular values
ii. Check orthogonality of columns of U and V
iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
iv. What is the rank of this matrix?
v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

d. Example #3: Image processing demonstration.