ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Oct. 30, 2024

1. Wave equation example

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

   a. Vibrations of string: $v = \sqrt{\dfrac{T}{\rho}}$, where T = tension in string, ρ = mass per unit length

   b. Typical BCs and ICs (two ICs because time problem is second order)

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

   c. Apply SOV method with $u(x, t) = X(x)T(t)$ to obtain

$$v^2 X''T = XT'' \quad \rightarrow \quad \frac{X''}{X} = \frac{T''}{v^2 T} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad T'' + v^2 \lambda T = 0$$

   d. The BCs $u(0, t) = 0 \rightarrow X(x) = 0$ and $u(L, t) = 0 \rightarrow X(L) = 0$ lead to the orthogonal spatial eigenfunctions

$$X_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

   e. The T problem is an initial value problem involving the Fourier equation. The possible solution forms are

$$T_n(t) = c_4 \cos\left(v\sqrt{\lambda_n} t\right) + c_5 \sin\left(v\sqrt{\lambda_n} t\right) = c_4 \cos\left(\frac{n\pi vt}{L}\right) + c_5 \sin\left(\frac{n\pi vt}{L}\right)$$

$$\text{or} \quad T_n(t) = c_4 e^{iv\sqrt{\lambda_n} t} + c_5 e^{-iv\sqrt{\lambda_n} t},$$

   but the trigonometric function form is more convenient since it involves only real quantities. Experience shows that this is true even though $t \rightarrow \infty$ (open "boundary").

   f. Each eigensolution to the full PDE with $A_n = c_2 c_4$ and $B_n = c_2 c_5$ has the form:

$$u_n(x,t) = X_n(x) T_n(t) = \left[A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

(*continued on next page*)

g. Full solution to the PDE for the general case (linear combination of eigensolutions):

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi vt}{L}\right)+B_n\sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

First time derivative (needed when second initial condition is applied):

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-A_n\frac{n\pi v}{L}\sin\left(\frac{n\pi vt}{L}\right)+B_n\frac{n\pi v}{L}\cos\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

h. Determination of coefficients in summation. Two sets of coefficients determined by two initial conditions (ICs):

Apply IC #1: $u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$

Multiply by *m*th eigenfunction and weighting function [$p(x)=1$ in this case] and integrate over interval of interest:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

As we have already seen, this leads to

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Apply IC #2: $\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)=\sum_{n=1}^{\infty}\left[-A_n\frac{n\pi v}{L}(0)+B_n\frac{n\pi v}{L}(1)\right]\sin\left(\frac{n\pi x}{L}\right)$

or $g(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi v}{L}\sin\left(\frac{n\pi x}{L}\right)$

Multiply by *m*th eigenfunction and weighting function [$p(x)=1$ again] and integrate over interval of interest to obtain

$$B_n=\frac{2}{L}\cdot\frac{L}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad\rightarrow\quad B_n=\frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

(*continued on next page*)

i. Full solution to the PDE is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi vt}{L}\right)\right]$$

with $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and $B_n = \frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

j. As with heat equation, the BCs typically determine the spatial eigenfunctions and the ICs typically determine the coefficients (using the orthogonality of the inner products). That is because the spatial problem is usually a BVP and the time problem is usually an IVP. EVPs are always BVPs.