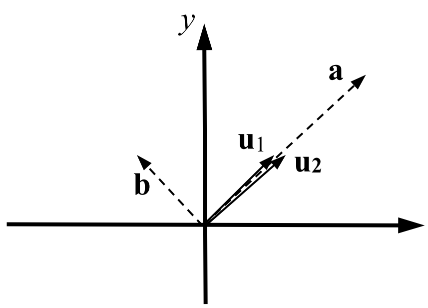
ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Aug. 30

1. Basic problems and computations in linear algebra: $A\mathbf{x} = \mathbf{b}$ [review]

   a. A, $\mathbf{x}$ given: geometric transformations (images, outputs from inputs)
   b. A, $\mathbf{b}$ given: system solution (inputs from outputs)
   c. Size & shape matter in defining the solution

2. A first look at solvability. When does $A\mathbf{x} = \mathbf{b}$ have a solution? When is the solution "iffy?" Start with an example: Express the 2-D vectors **a** and **b** defined below in terms of the nonorthogonal unit vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$.



$$\mathbf{a} = \hat{\mathbf{x}}2.82 + \hat{\mathbf{y}}2.80 \quad \text{and} \quad \mathbf{b} = -\hat{\mathbf{x}}0.80 + \hat{\mathbf{y}}0.80$$

$$\hat{\mathbf{u}}_1 = \hat{\mathbf{x}}\frac{1}{\sqrt{2}} + \hat{\mathbf{y}}\frac{1}{\sqrt{2}} \quad \text{and} \quad \hat{\mathbf{u}}_2 = \frac{\hat{\mathbf{x}}\left(\frac{1}{\sqrt{2}} + 0.01\right) + \hat{\mathbf{y}}\frac{1}{\sqrt{2}}}{\sqrt{\left[\left(\frac{1}{\sqrt{2}} + 0.01\right)^2 + \frac{1}{2}\right]}}$$

Expressing vectors like **a** and **b** in terms of orthonormal unit vectors like $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ is simple because the x and y-components can be determined by inspection. Finding the components in terms of a nonorthogonal set like $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ is more challenging, but it will help illustrate the concept of solvability (and the concepts of rank and eigenvectors later).

To express **a** in terms of $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$, we need to find its u_1-component a_1 and its u_2-component a_2. One approach is to find the dot product of **a** with $\hat{\mathbf{u}}_1$ and then with $\hat{\mathbf{u}}_2$:

$$\mathbf{a}\cdot\hat{\mathbf{u}}_1 = \left(\hat{\mathbf{u}}_1 a_1 + \hat{\mathbf{u}}_2 a_2\right)\cdot\hat{\mathbf{u}}_1 = a_1\left(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_1\right) + a_2\left(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2\right) = a_1 + a_2\left(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2\right)$$

and

$$\mathbf{a}\cdot\hat{\mathbf{u}}_2 = \left(\hat{\mathbf{u}}_1 a_1 + \hat{\mathbf{u}}_2 a_2\right)\cdot\hat{\mathbf{u}}_2 = a_1\left(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2\right) + a_2\left(\hat{\mathbf{u}}_2\cdot\hat{\mathbf{u}}_2\right) = a_1\left(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2\right) + a_2$$

These two equations in the unknowns a_1 and a_2 can be expressed in matrix form as

$$\begin{bmatrix} 1 & \hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}\cdot\hat{\mathbf{u}}_1 \\ \mathbf{a}\cdot\hat{\mathbf{u}}_2 \end{bmatrix}$$

The result is $\mathbf{a} = \hat{\mathbf{u}}_1 a_1 + \hat{\mathbf{u}}_2 a_2 = \hat{\mathbf{u}}_1 1.9598 + \hat{\mathbf{u}}_2 2.0142$, a vector with “nice,” small components that are of the same order as the *x* and *y*-components of **a**. (**a** is roughly in the same direction as $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$.)

Applying the same approach to express **b** in terms of $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ yields

$$\begin{bmatrix} 1 & \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \cdot \hat{\mathbf{u}}_1 \\ \mathbf{b} \cdot \hat{\mathbf{u}}_2 \end{bmatrix},$$

which gives the result $\mathbf{b} = -\hat{\mathbf{u}}_1 161.1314 + \hat{\mathbf{u}}_2 161.1353$.

The components are two orders of magnitude larger than the *x* and *y*-components, and they are almost exact negatives of each other. The differences do not appear until the sixth significant digit. (**b** is perpendicular to $\hat{\mathbf{u}}_1$ and almost perpendicular to $\hat{\mathbf{u}}_2$.)

Examine the parts of the matrix equations above:

$$\begin{bmatrix} 1 & \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 & 1 \end{bmatrix} = \begin{bmatrix} 1.00000000 & 0.99997535 \\ 0.99997535 & 1.00000000 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a} \cdot \hat{\mathbf{u}}_1 \\ \mathbf{a} \cdot \hat{\mathbf{u}}_2 \end{bmatrix} = \begin{bmatrix} 3.97394011 \\ 3.97394145 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{b} \cdot \hat{\mathbf{u}}_1 \\ \mathbf{b} \cdot \hat{\mathbf{u}}_2 \end{bmatrix} = \begin{bmatrix} 0.000000 \\ 0.007944 \end{bmatrix}$$

What if the basis vectors were collinear (and in this case not unit vectors)? For example, suppose that

$$\mathbf{u}_1 = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}2 \quad \text{and} \quad \mathbf{u}_2 = \hat{\mathbf{x}}10 + \hat{\mathbf{y}}10 \quad (\text{note that } \mathbf{u}_2 = 5\mathbf{u}_1)$$

Then:

$$\mathbf{a} \cdot \mathbf{u}_1 = (\mathbf{u}_1 a_1 + \mathbf{u}_2 a_2) \cdot \mathbf{u}_1 = a_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) + a_2 (\mathbf{u}_1 \cdot \mathbf{u}_2) = 8a_1 + 40a_2$$

and

$$\mathbf{a} \cdot \mathbf{u}_2 = (\mathbf{u}_1 a_1 + \mathbf{u}_2 a_2) \cdot \mathbf{u}_2 = a_1 (\mathbf{u}_1 \cdot \mathbf{u}_2) + a_2 (\mathbf{u}_2 \cdot \mathbf{u}_2) = 40a_1 + 200a_2$$

which becomes, in matrix form,

$$\begin{bmatrix} 8 & 40 \\ 40 & 200 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a} \cdot \hat{\mathbf{u}}_1 \\ \mathbf{a} \cdot \hat{\mathbf{u}}_2 \end{bmatrix} = \begin{bmatrix} 11.24 \\ 56.20 \end{bmatrix} \quad (\text{note that } 5 \times 11.24 = 56.20)$$

Does this system of equations have a solution? If so, how many?

Is the matrix singular or nonsingular? Does the matrix have an inverse? What is the determinant of the matrix?

3. Solution of $A\mathbf{x} = \mathbf{b}$ using the inverse. For an $N \times N$ (square) system, the following statements are equivalent for the purpose of determining the solvability of the problem.

    a. $A\mathbf{x} = \mathbf{b}$ has a unique solution
    b. A has a unique inverse (A^{-1})
    c. A is nonsingular
    d. $\det(A) = |A| \neq 0$
    e. A has full rank (i.e., $\text{rank}(A) = N$) – more on this later

4. Route to finding solutions of $A\mathbf{x} = \mathbf{b}$ (implicit inverse computation)

    a. Form augmented matrix $[A \mid \mathbf{b}]$
    b. Process: for augmented matrix, reduce (transform) a system to an easier-to-solve form
    c. $A\mathbf{x} = \mathbf{b}$ becomes $U\mathbf{x} = \mathbf{d}$ and solution ensues (U is upper triangular)
    d. Method: row reduction using elementary row operations (EROs); Gaussian elimination or Gauss-Jordan elimination
        i. Multiply a row (j) by a value (c)
        ii. Add a multiple (c) of one row (j) to another (k)
        iii. Interchange rows j and k

<u>Example Problems in Solving Systems of Linear Equations</u>

Prob. 1:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1 \\ 9x_1 - 2x_2 + x_3 &= -9 \\ 3x_1 + x_2 - 2x_3 &= -9 \end{aligned} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 3 & -1 & 1 & -1 \\ 9 & -2 & 1 & -9 \\ 3 & 1 & -2 & -9 \end{array}\right]$$

Prob. 2:

$$\begin{aligned} 3x - y - 2z &= 0 \\ -6x + 2y + 6z &= 4 \\ 2x + y + 6z &= 13 \end{aligned} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 3 & -1 & -2 & 0 \\ -6 & 2 & 6 & 4 \\ 2 & 1 & 6 & 13 \end{array}\right]$$

Prob. 3:

$$\begin{aligned} x + 2y - 5z &= 2 \\ 2x - 3y + 4z &= 4 \\ 4x + y - 6z &= 8 \end{aligned} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 1 & 2 & -5 & 2 \\ 2 & -3 & 4 & 4 \\ 4 & 1 & -6 & 8 \end{array}\right]$$