ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Sept. 2

1. Matrix equation from nonorthogonal unit vector example:

$$\begin{bmatrix} 8 & 40 \\ 40 & 200 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11.24 \\ 56.20 \end{bmatrix} \quad \text{(note that } 5 \times 11.24 = 56.20\text{)}$$

   Does this system of equations have a solution? If so, how many?

   Is the matrix singular or nonsingular? Does the matrix have an inverse? What is the determinant of the matrix?

2. Solution of $A\mathbf{x} = \mathbf{b}$ using the inverse: Start with the 1 × 1 (scalar) analogy: $ax = b$

   a. $1/a$ and a^{-1}: same thing but different perspective
   b. Key feature of a^{-1} required for solution
   c. How a^{-1} is used to get solution
   d. Roles of a and b in determining the number and nature of solution(s)

3. Solution of the $N \times N$ case: $A\mathbf{x} = \mathbf{b}$

   a. Inverse ("reciprocal") matrix A^{-1}
   b. Key feature of A^{-1} required for solution
   c. How A^{-1} is used to get solution
   d. How to find A^{-1}?

4. For an $N \times N$ (square) system, the following statements are equivalent for the purpose of determining the solvability of the problem.

   a. $A\mathbf{x} = \mathbf{b}$ has a unique solution
   b. A has a unique inverse (A^{-1})
   c. A is nonsingular
   d. $\det(A) = |A| \neq 0$
   e. A has full rank (i.e., $\text{rank}(A) = N$) – more on this later

Adapted from lecture outline by Prof. James Maneval, Fall 2019

5. Route to finding solutions of $A\mathbf{x} = \mathbf{b}$ (implicit inverse computation)

    a. Form augmented matrix $[A \mid \mathbf{b}]$
    b. Process: for augmented matrix, reduce (transform) a system to an easier-to-solve form
    c. $A\mathbf{x} = \mathbf{b}$ becomes $U\mathbf{x} = \mathbf{d}$ and solution ensues (U is upper triangular)
    d. Method: row reduction using elementary row operations (EROs); Gaussian elimination or Gauss-Jordan elimination
        i. Multiply a row (j) by a value (c)
        ii. Add a multiple (c) of one row (j) to another (k)
        iii. Interchange rows j and k

    Example Problems in Solving Systems of Linear Equations

    Prob. 1:

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1 \\ 9x_1 - 2x_2 + x_3 &= -9 \\ 3x_1 + x_2 - 2x_3 &= -9 \end{aligned} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 3 & -1 & 1 & -1 \\ 9 & -2 & 1 & -9 \\ 3 & 1 & -2 & -9 \end{array}\right]$$

    Probs. 2a and 2b:

$$\begin{aligned} x + 2y - 5z &= 2 \\ 2x - 3y + 4z &= 4 \\ 4x + y - 6z &= 8 \end{aligned} \qquad \text{Augmented matrix: } \left[\begin{array}{ccc|c} 1 & 2 & -5 & 2 \\ 2 & -3 & 4 & 4 \\ 4 & 1 & -6 & 8 \end{array}\right] \qquad \text{What if } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}?$$

6. Going further…

    a. Upper triangular (matrix entries below main diagonal are zero)
    b. Row-echelon form (upper triangular with 1s on main diagonal)
    c. Reduced row-echelon form (`rref` in *Matlab*; EROs until identity matrix is obtained or can't go any further)

7. Singular systems: rank and consistency

    a. Consistent: has at least one solution or infinitely many solutions (other numbers of solutions – e.g., 3 – are not possible). Example in 2-D: Solution space is a point or line
    b. Inconsistent: No solutions: Example in 2-D: Parallel lines – no intersecting point
    c. Rank: Max. no. of independent row vectors in a matrix (can use `rank` command in *Matlab*) – examples coming soon
    d. Relationship between rank and row-echelon (or reduced row-echelon) form