ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Oct. 2, 2024

1. New Topic: Ordinary differential equations (ODEs) – scope and coverage

    a. 2^{nd} order linear equations are some of the most important equations of mathematical physics
    b. Important special cases will be our focus
    c. Linear N^{th} order differential equation – general form:

$$a_n(x)\frac{d^n y}{dx^n}(x)+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_0(x)y=g(x)$$

        i. Coefficients can be constants (not functions of x) or variables (functions of x)
        ii. Homogeneous if $g(x) = 0$; otherwise, nonhomogeneous

    d. Initial value problems (IVPs) have initial conditions and unique solutions:

$$y(x_0)=y_0, \quad y'(x_0)=y_1, \quad \ldots \quad y^{(n-1)}(x_0)=y_{n-1} \qquad \text{(defined at } x = x_0\text{)}$$

    e. Boundary value problems (BVPs):

        i. Boundary conditions: Dependent variable and/or its derivatives defined at two or more points
        ii. Can have no solution, a unique solution, or many solutions
        iii. Special class of BVPs: eigenvalue problems (EVPs)
        iv. EVPs will be the main emphasis for this part of the course

2. Solution strategies

    a. Identify class of ODE
    b. Applicable solution structure (What should solution look like?)
    c. Corresponding solution method
    d. Starting point: homogenous solutions

3. Solvable BVP class #1: Constant coefficients

    a. Chapter 2 covers 1^{st} order with constant coefficients (skim)
    b. Section 3.3 covers 2^{nd} order with constant coefficients
    c. Proposed solution and consequence
    d. Cases and solution forms (special cases: Fourier equation $y''+a^2y=0$ and modified Fourier equation $y''-a^2y=0$)

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4. Solvable BVP class #2: Variable coefficients

   a. Some general forms considered (not an exhaustive list); primary focus is on first two, which are applicable to common PDEs:

      i. Cauchy-Euler equation: $ax^2y'' + bxy' + cy = 0$
      ii. Bessel equation: $x^2y'' + xy' + (x^2 - n^2)y = 0$ (also parametric and modified)
      iii. Legendre equation: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$
      iv. Airy equation: $y'' \pm a^2xy = 0$
      v. Chebyshev equation: $(1-x^2)y'' - xy' + a^2y = 0$

   b. Proposed solution and consequence
   c. Cases and solution forms (special functions)

5. Important theorems and concepts applicable to ODEs

   a. IVPs have unique solutions (not true for BVPs)
   b. Superposition principle; a linear combination of solutions to a homogeneous ODE over an interval is also a solution over the same interval
   c. Corollaries: 1) A constant multiple of a solution is also a solution; 2) homogeneous ODEs always possess at least the trivial solution $y = 0$
   d. Linearly independent vs. linearly dependent solutions (analogy to vectors)
   e. An N^{th} order homogeneous linear ODE has a fundamental set of N linear independent solutions. The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

   f. Nonhomogeneous ODEs: general solution = complementary solution + particular solution (complementary solution is the full solution set of the corresponding homogenous ODE)
   g. Superposition also applies to particular solutions: If y_{p1} is a solution of the DE with $g_1(x)$, y_{p2} is a solution with $g_2(x)$, etc., then $y_{p1} + y_{p2} + \ldots$ is a solution to

$$a_n(x)\frac{d^ny}{dx^n}(x) + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots$$

6. Solution of nonhomogeneous linear ODEs with constant coefficients (not emphasized here)

   a. You can guess or…
   b. Method of undetermined coefficients (Sec. 3.4 of Zill, 6th ed.) – doesn't work for all forcing functions
   c. Method of variation of parameters (Sec. 3.5 of Zill, 6th ed.) – more general & complicated
   d. Use the annihilator method (see web link) – annihilators do not always exist