ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Oct. 28, 2024

1. Interpretation of solution

$$u(x,t)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)e^{-kn^2\pi^2 t/L^2}$$

   a. How do we break it down?
   b. Does it satisfy BCs?
   c. What can we learn from it?
   d. Does it make sense? Behavior as $n \to \infty$ and $t \to \infty$
   e. Behavior of coefficients in infinite series as $n \to \infty$:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

      i. As $n \to \infty$, the eigenfunction $\sin(n\pi x/L)$ consists of a large number of cycles that span $x = 0$ to $x = L$.
      ii. If $f(x)$ is relatively smooth, then it is almost constant over each individual cycle of $\sin(n\pi x/L)$.
      iii. The integral from 0 to L can be decomposed into a sum of n different integrals over the individual cycles of $\sin(n\pi x/L)$.
      iv. Each of those integrals evaluates to almost zero because there are almost equal areas under the curve above and below the x-axis. Thus, $A_n \to 0$ as $n \to \infty$.

   f. *Matlab* simulation

2. Not always possible to find a solution with the SoV method; some PDE solutions are not separable. Which of the following PDEs can be solved via SOV, and which cannot?

   a. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$
   b. $a^2\dfrac{\partial^2 u}{\partial x^2} - g = \dfrac{\partial^2 u}{\partial t^2}$, where g is a constant
   c. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} + u$
   d. $a^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial u}{\partial t}$

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3. Wave equation example

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L\,, \quad t \ge 0$$

a. Vibrations of string: $v = \sqrt{\dfrac{T}{\rho}}$, where T = tension in string, ρ = mass per unit length

b. Typical BCs and ICs (two ICs because time problem is second order)

$$u(0,t) = 0\,, \quad u(L,t) = 0\,, \quad u(x,0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

c. Apply SOV method with $u(x, t) = X(x)T(t)$ to obtain

$$v^2 X''T = XT'' \quad \rightarrow \quad \frac{X''}{X} = \frac{T''}{v^2 T} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad T'' + v^2 \lambda T = 0$$

d. The BCs $u(0, t) = 0 \rightarrow X(x) = 0$ and $u(L, t) = 0 \rightarrow X(L) = 0$ lead to the orthogonal spatial eigenfunctions

$$X_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1,\, 2,\, 3,\, \ldots$$

e. The T problem is an initial value problem involving the Fourier equation. The possible solution forms are

$$T_n(t) = c_4 \cos\left(v\sqrt{\lambda_n}\, t\right) + c_5 \sin\left(v\sqrt{\lambda_n}\, t\right) = c_4 \cos\left(\frac{n\pi v t}{L}\right) + c_5 \sin\left(\frac{n\pi v t}{L}\right)$$

$$\text{or} \quad T_n(t) = c_4 e^{iv\sqrt{\lambda_n}\, t} + c_5 e^{-iv\sqrt{\lambda_n}\, t}\,,$$

but the one that uses trigonometric functions is more convenient since it involves only real quantities. Experience shows that this is true even though $t \rightarrow \infty$ (an open "boundary").

f. Each eigensolution to the full PDE with $A_n = c_2 c_4$ and $B_n = c_2 c_5$ has the form:

$$u_n(x,t) = X_n(x) T_n(t) = \left[A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

g. Full solution to the PDE for the general case:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$