ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Sept. 27, 2024

1. QR decomposition demo using *Matlab*.

2. The greatest factorization of them all, singular value decomposition (SVD)

   a. Very good at handling difficult systems in which the matrix is very close to singular (ill conditioned), which can happen with large data sets, measurement errors, and/or noisy data, to name just a few issues often encountered in real problems.
   b. Recommended over the normal equation for solving difficult overdetermined systems. Main disadvantages are more memory storage (an extra matrix) and sometimes it is slower.
   c. Can also be used for data compression (demo soon).
   d. Using the so-called "economy-sized" or "thin" decomposition, an $M \times N$ matrix A can be expressed in the product form (assuming $M > N$ or $M = N$ for now)

   $$A = U\Sigma V^H,$$

   where U is an $M \times N$ column-orthogonal matrix, Σ (sometimes labeled S) is an $N \times N$ diagonal matrix, V is an $N \times N$ orthogonal matrix, and H indicates complex conjugate transpose (V can be complex if A is complex):

   $$U = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{M\times N} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_N \end{bmatrix}_{N\times N} \qquad V = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}_{N\times N}$$

   where $\mathbf{u}_i{}^T\mathbf{u}_j = \delta_{ij}$ (orthogonal) and $\mathbf{v}_i{}^T\mathbf{v}_j = \delta_{ij}$ (orthogonal)

   e. In the full SVD, U is $M \times M$ and Σ is $M \times N$, but parts of U and Σ are not necessary for non-square ($M > N$) systems, hence the "economy-sized" decomposition.
   f. Matlab command (full SVD unless option is added): `[U,S,V] = svd(A)`
   g. The diagonal elements of Σ are called *singular values*. They are always real and either positive or zero, even if A has complex entries. They can repeat. Thus,

   $$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq \sigma_N .$$

   Zero singular values, if any, occupy the highest index numbers (i.e., up to and including σ_N)

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3. Singular value decomposition (SVD): Implications and properties

   a. $A = U\Sigma V^T = \sum_{j=1}^{N} \sigma_j \mathbf{u}_j \mathbf{v}_j^T$, where $\mathbf{u}_j \mathbf{v}_j^T$ is an outer product, each of which is $M \times N$.
      i. The outer products have rank = 1. (See HW #2 Prob. 7.)
      ii. A is a weighted sum of rank-1 $M \times N$ matrices.
      iii. The weights (σ_j) grow progressively smaller.
      iv. If any of the singular values are zero or too small to matter (lost in the noise, for example), then A can be represented by two compact sets of orthogonal vectors $\{\mathbf{u}_i\}_{i = 1 \text{ to } r}$ and $\{\mathbf{v}_i\}_{i = 1 \text{ to } r}$, where $r < N$.
      v. The summation terms for zero or nearly zero singular values can be ignored. (Example given in image processing demo later.)

   b. Use complex conjugate transpose for complex matrix A.

   c. U and V are both orthogonal ($U^{-1} = U^T$ and $V^{-1} = V^T$); thus,
      $A = U\Sigma V^T \quad \rightarrow \quad AV = U\Sigma \quad \rightarrow \quad A\mathbf{v}_i = \sigma_i \mathbf{u}_i$

   d. $A = U\Sigma V^T \quad \rightarrow \quad A^T = V\Sigma^T U^T \quad \rightarrow \quad A^T U = V\Sigma \quad \rightarrow \quad A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i$

   e. $A^T A = \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T \Sigma V^T$
      $\rightarrow \quad \left(A^T A\right)V = V\Sigma^T\Sigma \quad \rightarrow \quad \left(A^T A\right)\mathbf{v}_i = \sigma_i^2 \mathbf{v}_i$
      σ_i^2 are the eigenvalues of $A^T A$, and $\{\mathbf{v}_i\}_{i = 1 \text{ to } N}$ are the eigenvectors

   f. $AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma\Sigma^T U^T$
      $\rightarrow \quad \left(AA^T\right)U = U\Sigma\Sigma^T \quad \rightarrow \quad \left(AA^T\right)\mathbf{u}_i = \sigma_i^2 \mathbf{u}_i$
      σ_i^2 are also the eigenvalues of AA^T, and $\{\mathbf{u}_i\}_{i = 1 \text{ to } N}$ are the eigenvectors

   g. If A is symmetric, then $A^T A = AA^T = A^2$, so $\lambda_i^2 = \sigma_i^2 \quad \rightarrow \quad |\lambda_i| = |\sigma_i|$ (sign ambiguity)

4. Applications and examples

   a. To solve a system $A\mathbf{x} = \mathbf{b}$ (for overdetermined and square systems; underdetermined requires interpretation):

      $$U\Sigma V^T \mathbf{x} = \mathbf{b} \quad \rightarrow \quad \Sigma V^T \mathbf{x} = U^T \mathbf{b} \quad \rightarrow \quad V^T \mathbf{x} = \Sigma^{-1} U^T \mathbf{b} \quad \rightarrow \quad \mathbf{x} = V\Sigma^{-1} U^T \mathbf{b}$$

      Since Σ is diagonal (in the "economy-sized" decomposition),

      $$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\sigma_N \end{bmatrix}_{N \times N}$$

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b. Example #1: Compare eigenvalues to singular values of symmetric matrix A (use *Matlab* `eig` and `svd` commands):

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ 4 & 5 & 7 \end{bmatrix}$$

   i. Notice order of singular values
   ii. Check orthogonality of columns of U and V
   iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
   iv. What is the rank of this matrix?
   v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

c. Example #2: Compare eigenvalues to singular values of the singular matrix A:

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ 4 & 1 & -6 \end{bmatrix}$$

   i. Notice order of singular values
   ii. Check orthogonality of columns of U and V
   iii. Compare condition number (using *Matlab* command `cond`) to σ_1/σ_3
   iv. What is the rank of this matrix?
   v. What are the ranks of $\mathbf{u}_1\mathbf{v}_1^T$, $\mathbf{u}_2\mathbf{v}_2^T$, and $\mathbf{u}_3\mathbf{v}_3^T$?

d. Example #3: Image processing demonstration.