ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Oct. 25, 2024

1. Example application of separation of variables method: Heat equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{with} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x)$$

   a. Review: Application of SOV method results in two linked ODEs:

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda k T = 0$$

   b. Boundary conditions become

$$u(0,t) = X(0)T(t) = 0 \quad \to \quad X(0) = 0$$
$$u(L,t) = X(L)T(t) = 0 \quad \to \quad X(L) = 0$$

   c. The $X(x)$ problem is an S-L problem, and the $T(t)$ problem is an IVP with a first-order ODE:

$$X'' + \lambda X = 0 \quad \text{with} \quad X(0) = 0 \text{ and } X(L) = 0$$

$$T' + \lambda k T = 0 \quad \to \quad T(t) = c_3 e^{-k\lambda t}$$

   d. Since the X problem has closed boundaries and is an S-L problem (homogeneous BCs), the most convenient solution form is

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

   e. Apply BCs to obtain eigenfunctions and eigenvalues

$$X_n(x) = c_{2n} \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

   f. Each eigensolution to the full problem (x and t domains) has the form:

$$u_n(x,t) = X(x)T(t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\lambda_n t}$$

   The unknown coefficient in the $T(t)$ solution has been subsumed into the composite constant A_n.; that is, $A_n = c_{2n}\, c_3$.

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g. Each eigensolution u_n is a solution, but a linear combination of eigensolutions is also a solution. Full solution to the PDE for the general case:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$$

h. Determination of coefficients in summation:

Apply IC: $$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Multiply by the *m*th eigenfunction and the weighting function [$p(x) = 1$ in this case] and integrate over interval of interest:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

Because the $X(x)$ problem is an S-L problem, the eigenfunctions are orthogonal, and the inner products for $m \neq n$ are therefore zero. Thus,

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = A_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx = A_m \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2m\pi x}{L}\right)\right]dx = A_m\left(\frac{L}{2}\right)$$

$$\rightarrow \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

For some initial value functions $f(x)$, the integral could be evaluated analytically. In general or perhaps to save time, the integral would be evaluated numerically.

2. Interpretation of solution

   a. How do we break it down?
   b. Does it satisfy BCs?
   c. Does it make sense? Behavior as $t \rightarrow \infty$
   d. What can we learn from it?
   e. *Matlab* simulation