ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Sept. 23, 2024

1. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)

   a. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
   b. Orthogonal matrices are not usually symmetric, although 2 × 2 orthogonal matrices can be symmetric. (An orthogonal *and* symmetric matrix is I because such a matrix satisfies $A^{-1} = A^T = A$.)
   c. Important application: Various kinds of factorizations (also called decompositions) and diagonalizations, which can improve the efficiency of difficult computations and reveal conditioning (sensitivity to round-off errors) of matrices
   d. From MathWorks Help Center: "Orthogonal…matrices are desirable for numerical computation because they preserve length, preserve angles, and do not magnify errors."
   e. If A is orthogonal, then det(A) = ±1
   f. If A has complex elements, then A is called *unitary* if $A^{-1} = A^*$, where * indicates the Hermitian, or complex conjugate, transpose
   g. Example: Check that $A^{-1} = A^T$ and that the columns form an orthonormal set of vectors.

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

2. Diagonalization

   a. Express $N \times N$ matrix A as $A = PDP^{-1}$, where D is a diagonal matrix
   b. A does not have to be symmetric or orthogonal to be diagonalizable.
   c. Theorem: An $N \times N$ matrix A is diagonalizable iff A has N LI eigenvectors.
   d. Theorem: If $N \times N$ matrix A has N LI eigenvectors, then it is diagonalizable.
   e. Theorem: If $N \times N$ matrix A has N distinct eigenvalues, then it is diagonalizable (but it might be diagonalizable even if the eigenvalues are not distinct, that is, if some are repeated, as long as its eigenvectors are LI).
   f. Theorem: An $N \times N$ matrix A can be orthogonally diagonalized iff A is symmetric. (*Orthogonal diagonalization* means that P is orthogonal.)
   g. Many ways to diagonalize a matrix.

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h. One important example: Given an $N \times N$ matrix A with LI eigenvectors, since $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$, etc., then, using the definitions

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

where X is formed by making its columns equal to the eigenvectors, we obtain $AX = X\Lambda \rightarrow A = X\Lambda X^{-1}$. Thus, an $N \times N$ matrix A with LI eigenvectors can be diagonalized into its eigenvalues and eigenvectors.

i. Example: Attempt to diagonalize the following matrix:

$$A = \begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$$

Eigenvalues are 1 and 4. Determination of eigenvectors:

$$(A - \lambda_1 I)\mathbf{x}_1 = \mathbf{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} -5-1 & 9 & 0 \\ -6 & 10-1 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -6 & 9 & 0 \\ -6 & 9 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -6 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad -6x_{11} + 9x_{12} = 0 \quad \rightarrow \quad x_{12} = \frac{2}{3}x_{11} \quad \rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} -5-4 & 9 & 0 \\ -6 & 10-4 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -9 & 9 & 0 \\ -6 & 6 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} -9 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \quad -9x_{21} + 9x_{22} = 0 \quad \rightarrow \quad x_{22} = x_{21} \quad \rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus,

$$A = X\Lambda X^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

If the eigenvectors are normalized,

$$A = X\Lambda X^{-1} = \begin{bmatrix} 0.8321 & 0.7071 \\ 0.5547 & 0.7071 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0.8321 & 0.7071 \\ 0.5547 & 0.7071 \end{bmatrix}^{-1}$$

Note that X is not orthogonal because A is not symmetric.