ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Oct. 23, 2024

1. New major topic: Partial differential equations (PDEs). Applications:

    a. Heat transfer
    b. Diffusion
    c. Wave propagation (acoustic, mechanical, electromagnetic, etc.)
    d. Acoustics
    e. Highway traffic
    f. Many, many others

2. General form and classification (significance of classification most apparent in Chap. 16):

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

    a. Hyperbolic: $B^2-4AC>0$
    b. Parabolic: $B^2-4AC=0$
    c. Elliptic: $B^2-4AC<0$

3. Types of PDEs and conditions on which we will concentrate

    a. 1-D heat equation: $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \quad k\geq 0$ (parabolic)
    b. 2-D wave equation: $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ (hyperbolic)
    c. 2-D Laplace's equation: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (elliptic)
    d. All three PDEs above can be extended to more physical dimensions
    e. All problems have boundary conditions and initial conditions
    f. Many are eigenvalue problems
    g. Later: Numerical solutions to PDEs

4. Method of separation of variables (SOV)

    a. Overall solution assumed to be product of functions in each independent variable.
    b. Heat equation example:

Let $u(x,t)=X(x)T(t)$

(*continued on next page*)

Substitution leads to

$$k\frac{\partial^2 (XT)}{\partial x^2} = \frac{\partial (XT)}{\partial t} \quad \rightarrow \quad kX''T = XT' \quad \rightarrow \quad \frac{X''}{X} = \frac{T'}{kT},$$

where constant k is the thermal diffusivity. The goal is to make one side dependent on only one variable. In this case, the left-hand side depends only on x and the right-hand side only on t.

c. Introduce a new quantity, the separation constant:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

One view of why this works: Consider a third arbitrary independent variable; call it z. Take the derivative of the separated equation with respect to z:

$$\frac{\partial}{\partial z}\left(\frac{X''}{X}\right) = \frac{\partial}{\partial z}\left(\frac{T'}{kT}\right) = 0$$

Because both derivatives are zero, each side of the separated equation must be equal to a constant. Because the undifferentiated functions are equal, they must equal the same constant.

d. Result is two linked ODEs:

$$\frac{X''}{X} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad \frac{T'}{kT} = -\lambda \quad \rightarrow \quad T' + \lambda kT = 0$$

e. The solution to the ODE in one independent variable could be an infinite sum of eigenfunctions corresponding to different eigenvalues (superposition principle).

f. Two questions to be answered soon:

   i. Which ODE should be associated with the constants in the original PDE (e.g., the constant k in the heat equation example above)?
   ii. Why do we use λ instead of $-\lambda$ for the separation constant?

g. It is not always possible to find a solution with this method; some PDE solutions are not separable.