ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Nov. 22, 2024

1. Open boundaries in FD solution of wave equation

   a. Concept of one-way wave equation

$$v^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \quad\rightarrow\quad v^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0 \quad\rightarrow\quad \left(v^2\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)u=0 \quad\rightarrow\quad \left(v\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)u=0$$

   This is the product of two "one-way" wave equations:

$$v\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=0 \quad \text{and} \quad v\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0$$

   b. The first equation describes a leftward traveling wave (in −*x* direction), and the second equation describes a rightward traveling wave (in +*x* direction).

   Confirmation of first case:

   Let $u(x,t)=f(x+vt)$ ← leftward traveling wave with same shape as *f*(*x*)

   Substitute into one-way wave equation for leftward traveling waves:

$$\left(v\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)\left\{f(x+vt)\right\}=v\frac{\partial}{\partial x}\left\{f(x+vt)\right\}-\frac{\partial}{\partial t}\left\{f(x+vt)\right\}=0$$

   Define the new variable $p = x + vt$ so that the chain rule can be applied:

$$v\frac{\partial}{\partial p}\left\{f(p)\right\}\frac{\partial p}{\partial x}+\frac{\partial}{\partial p}\left\{f(p)\right\}\frac{\partial p}{\partial t}=0\,.$$

   Since

$$\frac{\partial p}{\partial x}=1 \qquad \text{and} \qquad \frac{\partial p}{\partial t}=v\,,$$

   then

$$v\frac{\partial}{\partial p}\left[f(p)\right](1)-\frac{\partial}{\partial p}\left[f(p)\right](v)=v\left\{\frac{\partial}{\partial p}\left[f(p)\right]-\frac{\partial}{\partial p}\left[f(p)\right]\right\}=0\,,$$

   which proves that $f(x + vt)$ is a solution.

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c. Apply the appropriate one-way wave equation at each boundary to obtain an open boundary condition (example: $x = a$ case, where waves should propagate out of space in the $-x$ direction):

$$\left(v\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)u=0 \quad \rightarrow \quad \left.\frac{\partial u}{\partial x}\right|_{x=a}=\frac{1}{v}\left.\frac{\partial u}{\partial t}\right|_{x=a}$$

d. FD form of one-way wave equation at boundary (centered in space at $i = 1$; centered in time at j)

$$\frac{u_{2,j}-u_{0,j}}{2\Delta x}=\frac{1}{v}\left(\frac{u_{1,j+1}-u_{1,j-1}}{2\Delta t}\right)$$

leads to, after solving for $u_{0,j}$,

$$u_{0,j}=u_{2,j}-\frac{\Delta x}{v_p\Delta t}\left(u_{1,j+1}-u_{1,j-1}\right)=u_{2,j}-\frac{1}{C}\left(u_{1,j+1}-u_{1,j-1}\right), \quad \text{where} \quad C=\frac{v\Delta t}{\Delta x}$$

e. Regular update equation applied at location $i = 1$:

$$u_{1,j+1}=C^2u_{2,j}+2\left(1-C^2\right)u_{1,j}+C^2u_{0,j}-u_{1,j-1}$$

f. Substitute expression for $u_{0,j}$ from one-way wave equation into regular update equation to obtain special update equation used only at the $x = a$ boundary:

$$u_{1,j+1}=\frac{2C^2}{1+C}u_{2,j}+2\left(1-C\right)u_{1,j}-\frac{1-C}{1+C}Cu_{1,j-1}, \quad \text{where} \quad C=\frac{v\Delta t}{\Delta x}$$

g. Similar update equation for use only at the $x = b$ boundary. Start with

$$\left.\frac{\partial u}{\partial x}\right|_{x=b}=-\frac{1}{v_p}\left.\frac{\partial u}{\partial t}\right|_{x=b}$$

to obtain

$$u_{N_x,j+1}=2\left(1-C\right)u_{N_x,j}+\frac{2C^2}{1+C}u_{N_x-1,j}-\frac{1-C}{1+C}u_{N_x,j-1}$$

h. Special case for first time step, $j = 0$. Start with initial condition applied at $x = a$:

$$g\left(a\right)=\left.\frac{\partial u}{\partial t}\right|_{t=0}\approx\frac{u\left(a,0+\Delta t\right)-u\left(a,0-\Delta t\right)}{2\Delta t}=\frac{u_{1,1}-u_{1,-1}}{2\Delta t}$$

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Substitute into special update equation for $i = 1$ to obtain special update equation for $i = 1$ ($x = a$) and $j = 0$:

$$u_{1,1} = C^2 u_{2,\,0} + \left(1 - C^2\right) u_{1,\,0} + \left(1 - C\right) \Delta t\, g\left(a\right)$$

i. Similar result for $j = 0$ and $i = N_x$ ($x = b$), the right-most boundary:

$$u_{N_x,1} = \left(1 - C^2\right) u_{N_x,\,0} + C^2 u_{N_x - 1,\,0} + \left(1 - C\right) \Delta t\, g\left(b\right)$$

j. Demonstration in next lab assignment

2. Next: Crank-Nicholson Method (an implicit method) applied to heat equation