**ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024**

**Lecture Outline for Monday, Oct. 21, 2024**

1. Orthogonality conditions on solutions to Sturm-Liouville problem (continued)

    a. The following relationship can be derived from the self-adjoint form of an ODE:

    $$(\lambda_m - \lambda_n)\int_a^b p(x)\,y_m(x)\,y_n(x)\,dx = r(b)\left[y_m(b)\,y_n'(b) - y_n(b)\,y_m'(b)\right] - r(a)\left[y_m(a)\,y_n'(a) - y_n(a)\,y_m'(a)\right]$$

    b. Note that the left-hand side includes the inner product. One implication of this result is that if $r(x) > 0$ everywhere, then the boundary conditions at $x = a$ and $x = b$ must be homogenous if the solutions $y_m$ and $y_n$ are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero. (See items #3 and #5 below.)

    Homogenous BCs at $x = a$ $\rightarrow$ $y_m(a)\,y_n'(a) - y_n(a)\,y_m'(a) = 0$

    Homogenous BCs at $x = b$ $\rightarrow$ $y_m(b)\,y_n'(b) - y_n(b)\,y_m'(b) = 0$

    c. Another implication is that $y_m$ and $y_n$ can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous.

2. Singular Sturm-Liouville problem

    a. Addresses cases when $r(x) > 0$ is not satisfied at one or both boundaries

    b. Right-hand side of equation in item 1a above is zero when:

        i. $r(a) = 0$ and $y_m(b)\,y_n'(b) - y_n(b)\,y_m'(b) = 0$
        ii. $r(b) = 0$ and $y_m(a)\,y_n'(a) - y_n(a)\,y_m'(a) = 0$
        iii. $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
        iv. $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)

    c. Warning: The solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ *provided that* $y_m(x)$ and $y_n(x)$ are bounded (i.e., do not go to $\pm\infty$) at the corresponding boundary.

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3. Homogenous boundary conditions

$$A_1 y(a) + B_1 y'(a) = 0$$

$$A_2 y(b) + B_2 y'(b) = 0$$

If the general BCs at $x = a$ are homogeneous, then

$$\begin{matrix} A_1 y_m(a) + B_1 y_m'(a) = 0 \\ A_1 y_n(a) + B_1 y_n'(a) = 0 \end{matrix} \quad \rightarrow \quad \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Can only be true for nonzero coefficient vector $[A_1\ B_1]^T$ if determinant of matrix is zero. That is,

$$y_m(a) y_n'(a) - y_n(a) y_m'(a) = 0$$

This relationship is also satisfied if either $A_1 = 0$ or $B_1 = 0$. For example, if $A_1 \neq 0$ and $B_1 = 0$, then $y_m(a) = 0$ and $y_n(a) = 0$, which still guarantees that $y_m(a) y_n'(a) - y_n(a) y_m'(a) = 0$.

Similar results for general BC at $x = b$. That is, if the BCs at $x = b$ are homogeneous, then

$$y_m(b) y_n'(b) - y_n(b) y_m'(b) = 0$$

4. Example: Recall the parametric Bessel's equation

$$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

Conversion to Sturm-Liouville equation in self-adjoint form yields $r(x) = x$, so $r(0) = 0$. We considered the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

General solution is

$$y(x) = c_1 J_0\left(\sqrt{\lambda} x\right) + c_2 Y_0\left(\sqrt{\lambda} x\right),$$

but this is a singular S-L problem because $r(0) = 0$. Also, because $Y_0(0) \rightarrow -\infty$, $Y_0\left(\sqrt{\lambda} x\right)$ is not a viable solution (it's not bounded). However, we can show that (for the boundary at $x = b = 1$)

$$y_m(1) y_n'(1) - y_n(1) y_m'(1) = (0) y_n'(1) - (0) y_m'(1) = 0$$

because the second BC $y(1) = 0$ applies to all solutions. The problem meets condition *i* in item 2b above. Thus, there are nontrivial, orthogonal solutions to this BVP.