ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Sept. 20, 2024

1. Eigensystem insights and expectations

   a. Simple eigensystem has *N* distinct eigenvalues, *N* linearly independent eigenvectors.
   b. $A\mathbf{x}_i = \lambda\mathbf{x}_i \quad \rightarrow \quad AX = X\Lambda$
   c. Eigenvalues of upper and lower-triangular matrices are the diagonal values.
   d. Eigenvalues of diagonal matrices are the diagonal values.
   e. Eigenvalues can repeat, and one or more can be zero.
   f. An eigenvector multiplied by a scalar constant is still an eigenvector:
   $A\left(k\mathbf{x}_i\right) = \lambda\left(k\mathbf{x}_i\right) \quad \rightarrow \quad kA\mathbf{x}_i = k\lambda\mathbf{x}_i \quad \rightarrow \quad A\mathbf{x}_i = \lambda\mathbf{x}_i$
   g. $A\mathbf{x}_i = \lambda\mathbf{x}_i \quad \rightarrow \quad \left(A - \lambda I\right)\mathbf{x}_i = \mathbf{0}$ can be used to find eigenvalues and eigenvectors.

2. Examples: Compute eigenvalues and eigenvectors of *A*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 8 & 4 \\ 0 & -3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

3. Some basic theorems

   a. If *A* is square with real entries, then any complex eigenvalues/eigenvectors come in conjugate pairs.
   b. If *A* is square, then 0 is an eigenvalue iff (if and only if) *A* is singular.
   c. det(*A*) = $\lambda_1\lambda_2\lambda_3\ldots\lambda_N$
   d. If *A* is nonsingular and λ is an eigenvalue, then $1/\lambda$ is an eigenvalue of A^{-1}; both eigenvalues have the same corresponding eigenvectors.
   e. Eigenvalues of upper/lower-triangular and diagonal matrices are the diagonal values.

4. Symmetric matrices ($A^T = A$) behave well
   a. Eigenvalues are real; all eigenvectors are linearly independent (LI)
   b. Distinct eigenvalues → orthogonal eigenvectors (also LI)
   c. Repeated eigenvalues → LI eigenvectors, but they might not be orthogonal
   d. Linearly independent ≠ orthogonal, but all orthogonal matrices have LI eigenvectors
   e. Symmetric matrices can be singular and therefore have at least one zero eigenvalue; even so, all eigenvectors are LI.
   f. Although the eigenvectors of symmetric matrices are all LI, a nonsingular symmetric matrix could still be ill-conditioned

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5. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)
    a. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each column vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
    b. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because such a matrix satisfies $A^{-1} = A^T = A$.)
    c. Example: Check that $A^{-1} = A^T$ and that each column is normalized

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

6. Where we are heading: *LU* and *QR* factorizations and the SVD