ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Nov. 20, 2024

1. Finite difference solution of the 1-D heat equation with Neumann BCs (continued):

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{with} \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb} \quad \text{for } t \geq 0,$$

where u_{xa} and u_{xb} are usually constants but could be time varying

a. One approach (for BC at $x = a$):

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \rightarrow \quad \frac{u(a+\Delta x,t)-u(a-\Delta x,t)}{2\Delta x} = \frac{u_{2,j}-u_{0,j}}{2\Delta x} \approx u_{xa}$$

Double-sized interval ($2\Delta x$) does not add significant error; less error than forward or backward difference with interval Δx.

b. Note that $u_{0,j}$ (located at $x = a - \Delta x$) is outside solution space. Use FD approximation of BC above to express $u_{0,j}$ in terms of quantities that exist:

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa}$$

c. Update equation that applies to interior points:

$$\text{general case: } u_{i,j+1} = C_1 u_{i+1,j} + C_2 u_{i,j} + C_3 u_{i-1,j}$$

$$\text{for } i = 1\text{: } u_{1,j+1} = C_1 u_{2,j} + C_2 u_{1,j} + C_3 u_{0,j}$$

$$\text{where } C_1 = C_3 = \frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_2 = 1 - 2\frac{c\Delta t}{\Delta x^2}$$

d. Substitute expression for $u_{0,j}$ found from BC into the general update equation evaluated at i =1:

$$\rightarrow \quad u_{1,j+1} = C_1 u_{2,j} + C_2 u_{1,j} + C_3\left(u_{2,j} - 2\Delta x\, u_{xa}\right)$$

$$\rightarrow \quad u_{1,j+1} = C_4 u_{2,j} + C_2 u_{1,j} - C_5 u_{xa}\,,$$

$$\text{where} \quad C_4 = C_1 + C_3 = \frac{2c\Delta t}{\Delta x^2} \quad \text{and} \quad C_5 = 2\Delta x C_3 = 2\Delta x\frac{c\Delta t}{\Delta x^2} = \frac{2c\Delta t}{\Delta x}$$

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e. Similar result for BC at $x = b$:

$$u_{N_x,j+1} = C_1\left(u_{N_x-1,j} + 2\Delta x\, u_{xb}\right) + C_2\, u_{N_x,j} + C_3\, u_{N_x-1,j}$$

$$\rightarrow \quad u_{N_x,j+1} = C_4\, u_{N_x-1,j} + C_2\, u_{N_x,j} + C_5\, u_{xb}$$

f. These two special update equations are applied *only* at the boundaries.

g. *Matlab* simulation

2. Alternate approach for handling Neumann BCs

   a. Add "fictional" solution space points at $i = 0$ and $i = N_x + 1$.

   b. Increase size of solution vector `u` by two (i.e., to $N_x + 2$); append solution values to beginning and end of vector. Could instead add two special variables to hold `u` at the exterior points (i.e., at $i = 0$ and $i = N_x + 1$).

   c. Update equations applied to end points (after interior points have been updated):

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa} \quad \text{and} \quad u_{N_x+1,j+1} = u_{N_x-1,j+1} + 2\Delta x\, u_{xb}$$