ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Nov. 1, 2024

1. Wave equation example (continued)

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

with BCs $u(0,t)=0$, $u(L,t)=0$, and ICs $u(x,0)=f(x)$, and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

a. Vibrations of string: $v = \sqrt{\frac{T}{\rho}}$, where T = tension in string, ρ = mass per unit length

b. Full solution to the PDE for the general case (linear combination of eigensolutions):

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi vt}{L}\right) + B_n \sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

c. First BC led to

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

d. First time derivative needed to apply second initial condition:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n \frac{n\pi v}{L}\sin\left(\frac{n\pi vt}{L}\right) + B_n \frac{n\pi v}{L}\cos\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Apply IC #2: $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty}\left[-A_n \frac{n\pi v}{L}(0) + B_n \frac{n\pi v}{L}(1)\right]\sin\left(\frac{n\pi x}{L}\right)$

or $g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi v}{L}\sin\left(\frac{n\pi x}{L}\right)$

Multiply by *m*th eigenfunction and weighting function [$p(x) = 1$ again] and integrate over interval of interest to obtain

$$B_n = \frac{2}{L}\cdot\frac{L}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \rightarrow \quad B_n = \frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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e. Full solution to the PDE is

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)+B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi vt}{L}\right)\right]$$

with $A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and $B_n=\frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

f. As with heat equation, the BCs typically determine the spatial eigenfunctions and the ICs typically determine the coefficients (using the orthogonality of the inner products) because the spatial problem is usually a BVP and the time problem is usually an IVP. EVPs are always BVPs.

2. Interpretation of wave equation solution

   a. What can we learn from it?
   b. Vibration modes and resonances; harmonically related
   c. *Matlab* simulation
   d. Standing waves vs. traveling waves. For example, consider the $g(x) = 0$ (or $B_n = 0$) case. Use the identity

$$\sin(a)\cos(b)=\frac{1}{2}\sin(a+b)+\frac{1}{2}\sin(a-b)$$

to obtain

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)=\frac{1}{2}\sin\left[\frac{n\pi}{L}(x+vt)\right]+\frac{1}{2}\sin\left[\frac{n\pi}{L}(x-vt)\right]$$

   e. Not modeled in this example:

      i. Acoustic coupling between strings and nearby non-fixed objects
      ii. Energy dissipation within string and nearby objects and due to air resistance
      iii. Scattering (reflections) from nearby objects
      iv. Presence of nearby objects, especially resonant cavities, which can affect sound perceived by listeners and can alter resonant frequencies somewhat
      v. Gravity