ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Oct. 18, 2024

1. Example of applying self-adjoint form: Solve the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

   a. Compare to general form of the Bessel equation $x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0 \rightarrow \nu = 0$
   b. Potential general solution form for closed boundaries:

$$y(x) = c_1 J_0\left(\sqrt{\lambda}x\right) + c_2 Y_0\left(\sqrt{\lambda}x\right) \quad \rightarrow \quad y'(x) = -c_1\sqrt{\lambda}J_1\left(\sqrt{\lambda}x\right) - c_2\sqrt{\lambda}Y_1\left(\sqrt{\lambda}x\right)$$

   See Eqn. (22) in Sec. 5.3.1 of the textbook (also applies to Bessel function of the second kind):

$$\frac{d}{dx}\left[x^{-\nu}J_\nu(x)\right] = -x^{-\nu}J_{\nu+1}(x) \quad \rightarrow \quad \frac{d}{dx}\left[x^{-0}J_0(x)\right] = -x^{-0}J_1(x) = -J_1(x)$$

   Apply BC #1:

$$y'(0) = 0 = -c_1\sqrt{\lambda}J_1(0) - c_2\sqrt{\lambda}Y_1(0) = -c_1\sqrt{\lambda}(0) - c_2\sqrt{\lambda}(-\infty)$$

   Since $Y_1(0) \rightarrow -\infty$ (as does $Y_0(0)$), $Y_0\left(\sqrt{\lambda}x\right)$ is not a viable solution. Apply BC #2:

$$y(1) = 0 = c_1 J_0\left(\sqrt{\lambda}\right)$$

   This implies that λ can only have values for which $\sqrt{\lambda_n} = r_n$, n = 1, 2, 3, …, where r_n are the roots (zeros) of J_0.

   c. First four roots of J_0: 2.4048, 5.5201, 8.6537, 11.7915 (see Table 5.3.1 in the textbook)

   d. Try evaluating inner product with and without $p(x) = x$; this is the focus of Lab #6:

$$\left\langle J_0\left(\sqrt{\lambda_m}x\right), J_0\left(\sqrt{\lambda_n}x\right)\right\rangle = \int_0^1 J_0\left(\sqrt{\lambda_m}x\right)J_0\left(\sqrt{\lambda_n}x\right)dx = ?$$

or

$$\left\langle J_0\left(\sqrt{\lambda_m}x\right), J_0\left(\sqrt{\lambda_n}x\right)\right\rangle = \int_0^1 xJ_0\left(\sqrt{\lambda_m}x\right)J_0\left(\sqrt{\lambda_n}x\right)dx = ?$$

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2. Orthogonality conditions on solutions to Sturm-Liouville problem

   a. Consider the Sturm-Liouville equation in self-adjoint form for two different eigenvalues

$$\frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right]+q(x)\,y_m+\lambda_m p(x)\,y_m=0$$

$$\frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right]+q(x)\,y_n+\lambda_n p(x)\,y_n=0$$

   b. Multiplying the first equation by y_n and the second by y_m, subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields

$$\begin{aligned}(\lambda_m-\lambda_n)\int_a^b p(x)\,y_m(x)\,y_n(x)\,dx = r(b)\left[y_m(b)\,y_n'(b)-y_n(b)\,y_m'(b)\right]\\ -r(a)\left[y_m(a)\,y_n'(a)-y_n(a)\,y_m'(a)\right]\end{aligned}$$

   c. Note that the left-hand side includes the inner product. One implication of this result is that if $r(x) > 0$ everywhere, then the boundary conditions at $x = a$ and $x = b$ must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero. (See items #3 and #5 below.)

   Homogenous BCs at $x = a$ → $y_m(a)\,y_n'(a)-y_n(a)\,y_m'(a)=0$
   Homogenous BCs at $x = b$ → $y_m(b)\,y_n'(b)-y_n(b)\,y_m'(b)=0$

   d. Another implication is that y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous.

3. Singular Sturm-Liouville problem

   a. Addresses cases when $r(x) > 0$ is not satisfied at one or both boundaries

   b. Right-hand side of equation in item 2b above is zero when:

      i. $r(a) = 0$ and $y_m(b)\,y_n'(b)-y_n(b)\,y_m'(b)=0$
      ii. $r(b) = 0$ and $y_m(a)\,y_n'(a)-y_n(a)\,y_m'(a)=0$
      iii. $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
      iv. $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)

   c. Warning: The solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ provided that $y_m(x)$ and $y_n(x)$ are bounded (i.e., do not go to $\pm\infty$) at the corresponding boundary.

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4. Example: Recall the parametric Bessel's equation

$$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

Conversion to Sturm-Liouville equation in self-adjoint form yields $r(x) = x$, so $r(0) = 0$. We considered the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

General solution is

$$y(x) = c_1 J_0\left(\sqrt{\lambda}x\right) + c_2 Y_0\left(\sqrt{\lambda}x\right),$$

but this is a singular S-L problem because $r(0) = 0$. Also, because $Y_0(0) \to -\infty$, $Y_0\left(\sqrt{\lambda}x\right)$ is not a viable solution (it's not bounded). However, we can show that

$$y_m(1)\, y_n'(1) - y_n(1)\, y_m'(1) = (0)\, y_n'(1) - (0)\, y_m'(1) = 0$$

because the second BC $y(1) = 0$ applies to all solutions. Thus, there are nontrivial, orthogonal solutions to this BVP.

5. Note that, if the BCs at $x = a$ are homogeneous, then

$$A_1 y_m(a) + B_1 y_m'(a) = 0 \quad \to \quad A_1 y_m(a) = -B_1 y_m'(a)$$

$$A_1 y_n(a) + B_1 y_n'(a) = 0 \quad \to \quad A_1 y_n(a) = -B_1 y_n'(a)$$

Dividing first equation by second (assuming that neither A_1 nor B_1 is zero) yields

$$\frac{y_m(a)}{y_n(a)} = \frac{y_m'(a)}{y_n'(a)} \quad \to \quad y_m(a)\, y_n'(a) - y_n(a)\, y_m'(a) = 0 .$$

Also satisfied if either $A_1 = 0$ or $B_1 = 0$. For example, if $A_1 \neq 0$ and $B_1 = 0$, then $y_m(a) = 0$ and $y_n(a) = 0$, which still guarantees that $y_m(a)\, y_n'(a) - y_n(a)\, y_m'(a) = 0$.

Alternative proof: BCs at $x = a$ for $m \neq n$ can be expressed as

$$\begin{aligned} A_1 y_m(a) + B_1 y_m'(a) &= 0 \\ A_1 y_n(a) + B_1 y_n'(a) &= 0 \end{aligned} \quad \to \quad \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Can only be true for nonzero coefficient vector $[A_1\ B_1]^T$ if determinant of matrix is zero.

Similar results for other general BC at $x = b$.