ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Nov. 18, 2024

1. FD solutions of wave equation: some computational considerations

   a. Accuracy generally improves as spatial step size Δx and/or time step size Δt is decreased, although not always (e.g., CFL condition is most accurate setting for Δt in explicit FD solution of wave equation)
   b. Grid dispersion: artificial change in velocity due to discretization in space/time
   c. Grid dissipation: artificial attenuation due to discretization in space/time
   d. Dispersion and dissipation both depend on frequency and/or pulse rise/fall time. Both can be reduced via small spatial step sizes.
   e. Finite precision of computer representation of numbers becomes a problem for very small step sizes

2. FD solutions in non-Cartesian coordinate systems

   a. Challenging due to more complicated expressions and variable step sizes
   b. Often simpler and/or more efficient to use Cartesian system and then apply a staircase approximation to irregular boundaries or interfaces between materials

3. Multiple materials in solution space:

   a. Boundary conditions might be necessary to account for interior material interfaces; these are in addition to the ones at the outer boundaries (i.e., those at $x = a$ and $x = b$).
   b. Some FD methods inherently account for interior boundaries (e.g., the "leap-frog" method for solving electric and magnetic fields in electromagnetic problems).

4. Finite difference solution of the 1-D heat equation with Neumann BCs:

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{with} \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb} \quad \text{for } t \geq 0,$$

   where u_{xa} and u_{xb} are usually constants but could be time varying

   a. Neumann BCs are often used to model insulation at boundaries.

   b. One approach (for BC at $x = a$):

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \rightarrow \quad \frac{u(a+\Delta x, t) - u(a-\Delta x, t)}{2\Delta x} = \frac{u_{2,j} - u_{0,j}}{2\Delta x} \approx u_{xa}$$

   Double-sized interval ($2\Delta x$) does not add significant error; less error than forward or backward difference with interval Δx.

(*continued on next page*)

c. Note that $u_{0,j}$ (located at $x = a - \Delta x$) is outside solution space. Use FD approximation of BC above to express $u_{0,j}$ in terms of quantities that exist:

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa}$$

d. Update equation that applies to interior points:

$$\text{general case: } u_{i,j+1} = C_1\, u_{i+1,j} + C_2\, u_{i,j} + C_3\, u_{i-1,j}$$

$$\text{for } i = 1\text{: } u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3\, u_{0,j}$$

$$\text{where } C_1 = C_3 = \frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_2 = 1 - 2\frac{c\Delta t}{\Delta x^2}$$

e. Substitute expression for $u_{0,j}$ into the general update equation evaluated at $i = 1$:

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa} \quad \text{and} \quad u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3\, u_{0,j}$$

$$\rightarrow \quad u_{1,j+1} = C_1\, u_{2,j} + C_2\, u_{1,j} + C_3\left(u_{2,j} - 2\Delta x\, u_{xa}\right)$$
$$\rightarrow \quad u_{1,j+1} = C_4\, u_{2,j} + C_2\, u_{1,j} - C_5\, u_{xa}\ ,$$

$$\text{where} \quad C_4 = C_1 + C_3 = \frac{2c\Delta t}{\Delta x^2} \quad \text{and} \quad C_5 = 2\Delta x C_3 = 2\Delta x\frac{c\Delta t}{\Delta x^2} = \frac{2c\Delta t}{\Delta x}$$

f. Similar result for BC at $x = b$:

$$u_{N_x,j+1} = C_1\left(u_{N_x-1,j} + 2\Delta x\, u_{xb}\right) + C_2\, u_{N_x,j} + C_3\, u_{N_x-1,j}$$

$$\rightarrow \quad u_{N_x,j+1} = C_4\, u_{N_x-1,j} + C_2\, u_{N_x,j} + C_5\, u_{xb}$$

g. These two special update equations are applied only at the boundaries.

5. Alternate approach for handling Neumann BCs

   a. Add "fictional" solution space points at $i = 0$ and $i = N_x + 1$.

   b. Increase size of solution vector `u` by two (i.e., to $N_x + 2$); append solution values to beginning and end of vector. Could instead add two special variables to hold `u` at the exterior points (i.e., at $i = 0$ and $i = N_x + 1$).

   c. Update equations applied to end points (after interior points have been updated):

$$u_{0,j} = u_{2,j} - 2\Delta x\, u_{xa} \quad \text{and} \quad u_{N_x+1,j+1} = u_{N_x-1,j+1} + 2\Delta x\, u_{xb}$$