ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Wednesday, Oct. 16, 2024

1. The SLIP is such a valuable set of properties that it is worth determining whether a given problem is a Sturm-Liouville problem. Examples:

    a. Is $y'' + \lambda y = 0$ a S-L equation?
    b. Is $y'' - \lambda y = 0$ a S-L equation?
    c. Is $x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$ a S-L equation?
    d. Is $y'' - \lambda xy = 0$ a S-L equation?
    e. Is $a(x) y'' + b(x) y' + c(x) y + \lambda d(x) y = 0$ a S-L equation?

2. To test whether an ODE is a Sturm-Liouville equation, convert it to self-adjoint form. That is, a second-order ODE of the form

$$a(x) y'' + b(x) y' + c(x) y + \lambda d(x) y = 0$$

can be converted to the equivalent form

$$\frac{d}{dx}\left[r(x) \frac{dy}{dx} \right] + q(x) y + \lambda p(x) y = 0$$

if $a(x) \neq 0$ everywhere.

3. Conversion to self-adjoint form:

    a. Compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx \right)$$

    b. Compute the variable coefficients of the S-L adjoint form:

$$r(x) = \mu(x) \qquad q(x) = \frac{c(x)}{a(x)} \mu(x) \qquad p(x) = \frac{d(x)}{a(x)} \mu(x)$$

    c. Verify that $r(x), p(x) > 0$ over the applicable interval of the solution.

    d. Note that $p(x)$ serves as the weighting function in the inner product that tests orthogonality.

(*continued on next page*)

4. Example #1: Are $y'' + \lambda y = 0$ and $y'' - \lambda y = 0$ S-L equations?

   a. First note that $a(x) = 1$, $b(x) = 0$, $c(x) = 0$, and $d(x) = \pm 1$. Then:

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp(0) = 1$$

$$\rightarrow \quad r(x) = \mu(x) = 1 \qquad q(x) = \frac{c(x)}{a(x)}\mu(x) = 0 \qquad p(x) = \frac{d(x)}{a(x)}\mu(x) = \pm 1$$

   b. Self-adjoint form of Fourier and modified Fourier equations:

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] \pm \lambda y = 0 \quad \text{or} \quad y'' \pm \lambda y = 0\text{, of course}$$

   c. Note that $r(x) > 0$ for both equations but that $p(x) > 0$ for the Fourier equation and $p(x) < 0$ for the modified Fourier equation. Thus, the Fourier equation satisfies the SLIP but not the modified Fourier equation. Also note that $p(x) = 1$ for the Fourier equation.

5. Example #2: Convert parametric Bessel's equation to Sturm-Liouville equation in self-adjoint form:

$$x^2 y'' + xy' + \left(\lambda x^2 - \nu^2\right) y = 0$$

   Identify the variable coefficients:

$$a(x) = x^2 \qquad b(x) = x \qquad c(x) = -\nu^2 \qquad d(x) = x^2$$

   Compute the integrating factor and other S-L equation factors:

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(\int \frac{x}{x^2} dx\right) = \exp\left(\int \frac{1}{x} dx\right) = e^{\ln x} = x$$

$$\rightarrow \quad r(x) = \mu(x) = x \qquad q(x) = \frac{-\nu^2}{x^2} x = \frac{-\nu^2}{x} \qquad p(x) = \frac{x^2}{x^2} x = x\,.$$

   Self-adjoint form of Bessel's equation:

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] - \frac{\nu^2}{x} y + \lambda x y = 0$$

   Significance: We now know the kernel $p(x)$ used in the inner product: $p(x) = x$.

(*continued on next page*)

6. Example of applying self-adjoint form: Solve the BVP

$$x^2 y'' + xy' + \lambda x^2 y = 0 \quad \text{with} \quad y'(0) = 0 \quad \text{and} \quad y(1) = 0$$

   a. Compare to general form of the Bessel equation $x^2 y'' + xy' + (\lambda x^2 - \nu^2) y = 0 \rightarrow \nu = 0$

   b. Potential general solution form for closed boundaries:

$$y(x) = c_1 J_0\left(\sqrt{\lambda} x\right) + c_2 Y_0\left(\sqrt{\lambda} x\right) \quad \rightarrow \quad y'(x) = -c_1 \sqrt{\lambda} J_1\left(\sqrt{\lambda} x\right) - c_2 \sqrt{\lambda} Y_1\left(\sqrt{\lambda} x\right)$$

   See Eqn. (22) in Sec. 5.3.1 of the textbook (also applies to Bessel function of the second kind):

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x) \quad \rightarrow \quad \frac{d}{dx}\left[x^{-0} J_0(x)\right] = -x^{-0} J_1(x) = -J_1(x)$$

   Apply BC #1:

$$y'(0) = 0 = -c_1 \sqrt{\lambda} J_1(0) - c_2 \sqrt{\lambda} Y_1(0) = -c_1 \sqrt{\lambda}(0) - c_2 \sqrt{\lambda}(-\infty)$$

   Since $Y_1(0) \rightarrow -\infty$ (as does $Y_0(0)$), $Y_0\left(\sqrt{\lambda} x\right)$ is not a viable solution. Apply BC #2:

$$y(1) = 0 = c_1 J_0\left(\sqrt{\lambda}\right)$$

   This implies that λ can only have values for which $\sqrt{\lambda_n} = r_n$, n = 1, 2, 3, …, where r_n are the roots (zeros) of J_0.

   c. First four roots of J_0: 2.4048, 5.5201, 8.6537, 11.7915 (see Table 5.3.1 in the textbook)

   d. Try evaluating inner product with and without $p(x) = x$; this is the focus of Lab #6:

$$\left\langle J_0\left(\sqrt{\lambda_m} x\right), J_0\left(\sqrt{\lambda_n} x\right)\right\rangle = \int_0^1 J_0\left(\sqrt{\lambda_m} x\right) J_0\left(\sqrt{\lambda_n} x\right) dx = ?$$

   or

$$\left\langle J_0\left(\sqrt{\lambda_m} x\right), J_0\left(\sqrt{\lambda_n} x\right)\right\rangle = \int_0^1 x J_0\left(\sqrt{\lambda_m} x\right) J_0\left(\sqrt{\lambda_n} x\right) dx = ?$$