ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Friday, Sept. 13

1. Introduction to eigenvalues, eigenvectors, and eigensystems. Start with example:

   a. $A\mathbf{x} = \mathbf{b} \quad \rightarrow \quad A\left(\sum_{i=1}^{N} a_i \mathbf{x}_i\right) = \sum_{i=1}^{N} \mathbf{b}_i \quad \rightarrow \quad \sum_{i=1}^{N} A a_i \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{b}_i$,
   where $\mathbf{x} = \sum_{i=1}^{N} a_i \mathbf{x}_i$ and $A a_i \mathbf{x}_i = \mathbf{b}_i$. The basis vectors $\{\mathbf{x}_i\}$ can often be arbitrarily chosen

   b. What if, for example, $\mathbf{x} = \sum_{i=1}^{N} a_i \hat{\mathbf{e}}_i$, where $\hat{\mathbf{e}}_i = \left[0 \cdots 0 \quad 1 \quad 0 \cdots 0\right]^T$ (a 1 in the *i*th location)?

   c. $\mathbf{b}_i$ might not be in the same direction as $\hat{\mathbf{e}}_i$ (rotation and maybe elongation/contraction)

   d. If $\mathbf{b}_i$ and $\hat{\mathbf{e}}_i$ have the same direction, then $\hat{\mathbf{e}}_i$ is an eigenvector.

   e. In general, if $\mathbf{x}_i$ and $\mathbf{b}_i$ have the same direction, then $\mathbf{x}_i$ is an eigenvector.

2. Eigensystem insights and expectations

   a. Simple eigensystem has *N* distinct eigenvalues, *N* linearly independent eigenvectors

   b. $A\mathbf{x}_i = \lambda \mathbf{x}_i \quad \rightarrow \quad AX = X\Lambda$

   c. Eigenvalues of upper and lower-triangular matrices are the diagonal values

   d. Eigenvalues of diagonal matrices are the diagonal values

   e. $A\mathbf{x}_i = \lambda \mathbf{x}_i \quad \rightarrow \quad (A - \lambda I)\mathbf{x}_i = \mathbf{0}$ can be used to find eigenvalues and eigenvectors.

3. Example: Compute eigenvalues and eigenvectors of *A*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

4. Some basic theorems

   a. If *A* is square with real entries, then any complex eigenvalues/eigenvectors come in conjugate pairs.

   b. If *A* is square, then 0 is an eigenvalue only iff *A* is singular.

   c. $\det(A) = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_N$

   d. If *A* is nonsingular and λ is an eigenvalue, then $1/\lambda$ is an eigenvalue of A^{-1}; both eigenvalues have the same corresponding eigenvectors.

   e. Eigenvalues of upper/lower-triangular and diagonal matrices are the diagonal values.

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5. Important special cases

   a. Symmetric matrices ($A^T = A$) behave well
      i. Eigenvalues are real; all eigenvectors are linearly independent (LI)
      ii. Distinct eigenvalues → orthogonal eigenvectors (also LI)
      iii. Repeated eigenvalues → LI eigenvectors but might not be orthogonal
      iv. Linearly independent ≠ orthogonal
      v. Symmetric matrices can be singular and therefore have at least one zero eigenvalue; even so, all eigenvectors are LI.
   b. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)
      i. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
      ii. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because a matrix that is both satisfies $A^{-1} = A^T = A$.)
      iii. Example: Check that $A^{-1} = A^T$ and that each column is normalized

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

6. Where we are heading: *LU* and *QR* factorizations and the SVD