ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Oct. 11, 2024

1. Important general form of ODE: the regular Sturm-Liouville equation

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right]+q(x)y+\lambda p(x)y=0$$

subject to homogeneous boundary conditions defined over interval [a, b]:

$$A_1 y(a)+B_1 y'(a)=0, \quad A_1 \text{ and } B_1 \text{ not both zero}$$
$$A_2 y(b)+B_2 y'(b)=0, \quad A_2 \text{ and } B_2 \text{ not both zero}$$

   a. some textbooks use $p(x)$ for $r(x)$ and $w(x)$ for $p(x)$
   b. second-order with variable coefficients
   c. form above is called "self-adjoint" form
   d. λ is a parameter in the problem (eigenvalue)
   e. must have $r(x)$, $p(x) > 0$ over interval of solution
   f. importance for our purposes:
      i. guarantees orthogonality, completeness, representation (see supplemental reading "Sturm-Liouville Problems: Eigenfunction Orthogonality")
      ii. function $p(x)$ defines kernel (weight function) for inner product

2. The "Sturm-Liouville Insurance Policy" (SLIP)

   a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
   b. There is an infinity of eigenvalues.
   c. There is a smallest but not a largest eigenvalue.
   d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$).
   e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
   f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$ over the interval [a, b]. That is,

$$\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x)\, y_n(x) p(x)\, dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

   g. The set of solutions to an S-L problem are complete in that the set forms a basis for the space of square-integrable functions over the interval [a, b].

$$f(x)=\sum_{n=1}^{\infty} a_n y_n(x) \quad \text{with} \quad a_n = \frac{\langle f(x), y_n(x)\rangle}{\langle y_n(x), y_n(x)\rangle}$$

(*continued on next page*)

Proof:

Multiply both sides of weighted sum expression by $y_m(x)$ and evaluate the inner product. Don't forget the weight function $p(x)$:

$$\int_a^b f(x)\, y_m(x)\, p(x)\, dx = \sum_{n=1}^{\infty} a_n \int_a^b y_m(x)\, y_n(x)\, p(x)\, dx$$

Because of the SLIP, we know that the eigenfunctions $y_n(x)$ are orthogonal. Thus,

$$\int_a^b y_m(x)\, y_m(x)\, p(x)\, dx = 0 \text{ for } m \neq n,$$

so

$$\int_a^b f(x)\, y_m(x)\, p(x)\, dx = a_m \int_a^b y_m(x)\, y_m(x)\, p(x)\, dx$$

Expressed in inner product notation,

$$\left\langle f(x), y_m(x) \right\rangle = a_m \left\langle y_m(x), y_m(x) \right\rangle \quad \rightarrow \quad a_m = \frac{\left\langle f(x), y_m(x) \right\rangle}{\left\langle y_m(x), y_m(x) \right\rangle}$$

The expression for a_n on the previous page follows after changing the index from m to n.

3. The SLIP is such a valuable set of properties that it is worth determining whether a given problem is a Sturm-Liouville problem. Examples:

   a. Is $y'' + \lambda y = 0$ a S-L equation?
   b. Is $y'' - \lambda y = 0$ a S-L equation?
   c. Is $y'' - \lambda x y = 0$ a S-L equation?
   d. Is $x^2 y'' + x y' + \left(\lambda x^2 - \nu^2\right) y = 0$ a S-L equation?
   e. Is $a(x)\, y'' + b(x)\, y' + c(x)\, y + \lambda d(x)\, y = 0$ a S-L equation?

4. To test whether an ODE is a Sturm-Liouville equation, convert it to self-adjoint form. That is, a second-order ODE of the form

$$a(x)\, y'' + b(x)\, y' + c(x)\, y + \lambda d(x)\, y = 0$$

   can be converted to the equivalent form

$$\frac{d}{dx}\left[r(x) \frac{dy}{dx} \right] + q(x)\, y + \lambda p(x)\, y = 0$$

   if $a(x) \neq 0$ everywhere.