ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Lecture Outline for Monday, Nov. 11, 2024

1. Finite difference approximation of second derivative:

$$\frac{d^2 f(x_o)}{dx^2} \approx \frac{f'\left(x_o + \frac{\Delta x}{2}\right) - f'\left(x_o - \frac{\Delta x}{2}\right)}{\Delta x}$$

$$\approx \frac{\left[\frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}\right] - \left[\frac{f(x_o) - f(x_o - \Delta x)}{\Delta x}\right]}{\Delta x}$$

$$= \frac{f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x)}{\Delta x^2}$$

2. Application: Finite difference solution of the heat equation

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{where } c = \text{thermal diffusivity}$$

   a. Questions:

      i. What do finite difference approximations look like when there is more than one independent variable? At what points in space and time are they centered?
      ii. How many solution points (in x and in t) do we select?
      iii. What do we do about the boundaries?

   b. Finite difference approximations of partial derivatives (hold undifferentiated variable constant)

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \quad \text{and} \quad \frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

   c. Note that the x-derivative is approximated using a centered difference and the time derivative by a forward difference. We will see why very soon.

   d. Heat equation expressed using finite differences

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

(*continued on next page*)

e. Define discrete points in space and time at which the dependent variable u is calculated. The arrays of points are called spatial and time *grids* or *meshes*.
   i. Solution space is along x-axis between boundaries $x = a$ and $x = b$.
   ii. Calculation time begins at $t = 0$.
   iii. Space and time are discretized into N_x and N_t points, respectively:

$$x_i = a + (i-1)\Delta x, \quad i = 1, 2, 3, \ldots, N_x \quad \text{where} \quad \Delta x = \frac{b-a}{N_x - 1}$$

$$t_j = j\Delta t, \quad j = 0, 1, 2, 3, \ldots, (N_t - 1)$$

f. In general, the smaller Δx and Δt are, the greater the accuracy of the finite difference solution. However, there is a constraint on Δt (examined soon).

g. Finite difference (FD) subscript notation:

$$u(x,t) = u_{i,j} \qquad u(x+\Delta x,t) = u_{i+1,j} \qquad u(x-\Delta x,t) = u_{i-1,j} \qquad u(x,t+\Delta t) = u_{i,j+1}$$

$$c\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

$$\rightarrow \quad c\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}$$

h. Four of the five terms in FD form of equation are defined at time t (index j), but one is defined at time $t + \Delta t$ (index $j + 1$). Isolate that term on the left-hand side and move the rest to the right-hand side to form an *update equation*:

$$u_{i,j+1} - u_{i,j} = \frac{c\Delta t}{\Delta x^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] \quad \rightarrow \quad u_{i,j+1} = \frac{c\Delta t}{\Delta x^2}u_{i+1,j} + \left(1 - 2\frac{c\Delta t}{\Delta x^2}\right)u_{i,j} + \frac{c\Delta t}{\Delta x^2}u_{i-1,j}$$

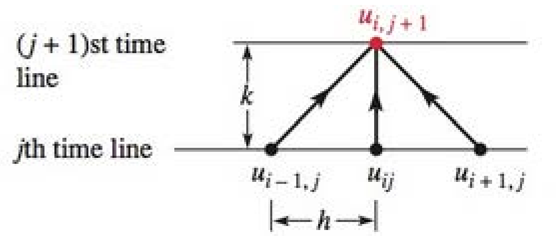
i. This is an *explicit* FD method. The newest value of u at location i depends only on previous values and no values at other locations at the new time. That is, there is only one term at time index $j + 1$. A system of simultaneous equations is not required to find u everywhere.

j. To improve computational efficiency (i.e., to minimize floating-point operations):
   i. Group like terms (for particular space and time indices) together.
   ii. Pre-calculate the coefficients and store them.

$$u_{i,j+1} = c_1 u_{i+1,j} + c_2 u_{i,j} + c_3 u_{i-1,j}, \qquad \text{where} \quad c_1 = c_3 = \frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad c_2 = 1 - 2\frac{c\Delta t}{\Delta x^2}$$

(*continued on next page*)

k. "Stencil" for the explicit FD solution of the heat equation. Next value of *u* depends only on the most recent values of *u* at the same and adjacent locations.



Source: Fig. 16.2.2 of D. G. Zill, *Advanced Engineering Mathematics*, 6th ed., Jones and Bartlett Learning, 2016.