ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Homework Assignment #7 – do not submit

***Ungraded Problems*:**

The following problems will not be graded, but you should attempt to solve them on your own and then check the solutions. Do not give up too quickly if you struggle with one or more of them. Move on to a different problem and then come back to the difficult one after a few hours.

1. Consider the Laplace's equation problem below with the indicated Neumann boundary conditions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le a \quad \text{and} \quad 0 \le y \le b$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0, \quad 0 < x < a \qquad \text{and} \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = g(y), \quad 0 < y < b.$$

The separation-of-variables solution is

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right),$$

where the constant A_0 is indeterminate (can have any value) in the absence of further information and where

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g(y)\cos\left(\frac{n\pi y}{b}\right) dy, \qquad n = 1, 2, 3, \ldots$$

It turns out that a necessary condition for a solution u to exist is that the function $g(y)$ must satisfy

$$\int_0^b g(y)\, dy = 0.$$

This is sometimes called a *compatibility condition*. Explain the compatibility condition on physical grounds and then prove it mathematically using the information above.
Hint: Consider that the three homogeneous boundary conditions imply perfect insulation on three sides. Also consider that Laplace's equation yields a steady-state solution to the heat distribution problem; that is, heat is not moving through the space.

(*continued on next page*)

2. [adapted from Prob. 5 in Sec. 13.5 of D. G. Zill, *Advanced Engineering Mathematics*, 6th ed.] Solve the Laplace's equation problem given below. You do not have to evaluate the integral to obtain a closed-form expression for the coefficients in the solution; that is, you may leave the formula for the coefficients expressed as an integral. Use the integral to find numerical values for the first three nonzero coefficients. You may use mathematical analysis software such as *Matlab* or *Mathematica* to perform numerical integrations. You may also use your calculator if it has that capability. *Hint*: The even coefficients are zero.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1$$

$$u(0, y) = 0, \quad u(1, y) = 1 - y, \quad \text{for } 0 < y < 1 \quad \text{and} \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0, \quad \text{for } 0 < x < 1.$$