ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Homework Assignment #6 – due via Moodle at 11:59 pm on Tuesday, Dec. 10

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work. You may use *Matlab* or other software to check your work.

It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

1. Derive an explicit finite difference update equation for the numerical solution of the following modified wave equation, which models a medium that offers resistance proportional to the instantaneous transverse velocity of the disturbance.

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t}, \qquad 0 \le x \le L, \quad 0 \le \beta \le 1, \quad \text{and} \quad t \ge 0$$

The FD approximations must all be spatially and temporally centered. (Use a $2\Delta t$ interval for the first-order time derivative.) Use the spatial and temporal indices i and n, respectively, defined by $x = (i-1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, and $t = n\,\Delta t$ for $n = 0, 1, 2, \ldots, N_t$, where N_x and N_t are the numbers of locations and time steps, respectively. Group like terms in the update equation so that the number of floating-point operations is minimized. You do not have to find the special update equation for the $n = 0$ (corresponding to $t = 0$) case.

2. Derive an explicit finite difference update equation for the numerical solution of the following modified wave equation, which models a string initially at rest on the x-axis but that is allowed to fall under its own weight for $t > 0$.

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

Variable $u(x, t)$ is the vertical displacement of the string, and the constant g is the acceleration of gravity. The FD approximations must use spatially and temporally centered differences. Use space and time grids defined by $x = (i-1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of spatial points, and $t = (n-1)\Delta t$ for $n = 1, 2, 3, \ldots, N_t$, where N_t is the total number of time steps. Group like terms in the update equation to minimize the number of floating-point operations. You do not have to find the special update equation for the $t = 0$ case.

(*continued on next page*)

3. A Robin boundary condition incorporates both the dependent variable and its first spatial derivative; it can be thought of as a linear combination of the Dirichlet and Neumann boundary conditions. A common application in heat equation problems arises when an object is in contact with a medium (such as the ground) that is held at a constant temperature. The transfer of heat is proportional to the difference in temperature between the end of the object at the boundary and the temperature u_m of the surrounding medium. If the boundary is located at $x = a$, the Robin boundary condition can be expressed as

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = h\left[u(a,t) - u_m\right] \quad \rightarrow \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} - h\,u(a,t) = -h\,u_m, \quad h > 0,$$

where h is a constant and where the expression on the right places the derivative and the undifferentiated dependent variable to the left of the equal sign and the constant to the right. Note that the boundary condition is nonhomogeneous if $u_m \neq 0$. Show that the modified update equation that must be applied at the boundary at $x = a$ to incorporate the Robin boundary condition into the explicit finite difference solution is given by

$$u_{1,\,j+1} = C_1\,u_{2,\,j} + C_2\,u_{1,\,j} + C_3\,u_m,$$

where

$$C_1 = \frac{2c\Delta t}{\Delta x^2}, \quad C_2 = 1 - \frac{2c\Delta t}{\Delta x^2} - \frac{2hc\Delta t}{\Delta x}, \quad \text{and} \quad C_3 = \frac{2hc\Delta t}{\Delta x}.$$

The solution space is discretized as $x = a + (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of points. The time index is j, where $j = 0$ corresponds to $t = 0$. Center the finite difference at location index $i = 1$, which corresponds to $x = a$. You do not have to find the modified update equation that is applicable at the other boundary (at $x = b$).

4. Repeat the previous problem for the Crank-Nicholson method described in Sec. 16.2 of the textbook (D. G. Zill, *Advanced Engineering Mathematics*, 6$^{\text{th}}$ ed.). Show that, to incorporate the Robin boundary condition at $x = a$, the modified first equation in the system of equations used to compute u at each time step is given by

$$-(\alpha + 2\Delta x h)u_{1,\,j+1} + 2u_{2,\,j+1} = (\beta + 2\Delta x h)u_{1,\,j} - 2u_{2,\,j} - 4\Delta x h\,u_m,$$

where

$$\alpha = 2\left(1 + \frac{\Delta x^2}{c\Delta t}\right) \quad \text{and} \quad \beta = 2\left(1 - \frac{\Delta x^2}{c\Delta t}\right).$$

Use the same space and time discretization scheme as in the previous problem. You do not have to find the modified equation that is applicable at the other boundary (at $x = b$).

(*continued on next page*)

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Derive an explicit finite difference update equation for the numerical solution of the two-dimensional heat equation problem given by

$$c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}, \qquad 0 \le x \le a, \quad 0 \le y \le b, \quad \text{and} \quad t \ge 0,$$

where c is the thermal diffusivity of the material. The spatial locations within the solution space are defined by $x = (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$ and $y = (j - 1)\Delta x$ for $j = 1, 2, 3, \ldots, N_y$, where N_x and N_y are the total number of discrete points in the x and y directions, respectively. The time variable is indexed as $t = (n - 1)\Delta t$ for $n = 1, 2, 3, \ldots, N_t$, where N_t is the total number of time steps. Group like terms in the update equation so that the number of floating-point operations is minimized. Use the derivation at the beginning of Sec. 16.2 of the textbook (D. G. Zill, *Advanced Engineering Mathematics*, 6th ed.) as a guide, and use the following constants to simplify the update equation:

$$C_x = \frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_y = \frac{c\Delta t}{\Delta y^2}.$$

2. Refer to the lecture notes "Open Boundaries in the Finite Difference Solution of the 1-D Wave Equation." The special update equations derived in the notes that are applied at the boundaries of the solution space are:

$$u_{1,\,j+1} = \frac{2C^2}{1+C}u_{2,\,j} + 2(1-C)u_{1,\,j} - \frac{1-C}{1+C}u_{1,\,j-1}, \quad \text{for } i = 1$$

$$u_{N_x,\,j+1} = 2(1-C)u_{N_x,\,j} + \frac{2C^2}{1+C}u_{N_x-1,\,j} - \frac{1-C}{1+C}u_{N_x,\,j-1}, \quad \text{for } i = N_x, \qquad \text{where} \quad C = \frac{v_p\Delta t}{\Delta x}.$$

Consider the case when Δt is set to the maximum possible value that satisfies the Courant-Friedrichs-Lewy (CFL) condition for stability. Given that these two boundary conditions are based on the one-way wave equations, explain why the resulting update equations obtained at the limit of the CFL condition make sense.

(*continued on next page*)

3. Neumann boundary conditions can be incorporated into the solution of the one-dimensional heat equation problem using the Crank-Nicholson method. Assume that the spatial domain extends from $x = a$ to $x = b$ and that the boundary conditions are given by

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb}\,,$$

where u_{xa} and u_{xb} are constants. The solution space is discretized as $x = a + (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of points. The time index is j, where $j = 0$ corresponds to $t = 0$. Show that the first and last equations in the system of equations needed to compute u at each time step with the boundary conditions shown above are given by

$$-\alpha u_{1,\,j+1} + 2u_{2,\,j+1} = \beta u_{1,\,j} - 2u_{2,\,j} + 4\Delta x\, u_{xa} \quad \text{and} \quad 2u_{N_x-1,\,j+1} - \alpha u_{N_x,\,j+1} = -2u_{N_x-1,\,j} + \beta u_{N_x,\,j} - 4\Delta x\, u_{xb}\,,$$

where

$$\alpha = 2\left(1 + \frac{\Delta x^2}{c\Delta t}\right) \quad \text{and} \quad \beta = 2\left(1 - \frac{\Delta x^2}{c\Delta t}\right).$$