ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Homework Assignment #5 – due via Moodle at 5:00 pm on Friday, Nov. 22, 2024

***Instructions, notes, and hints*:**

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

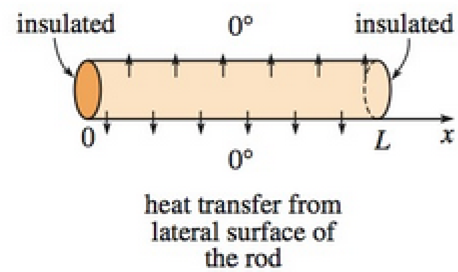
It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

1. [adapted from Prob. 5 in Sec. 13.3 of D. G. Zill, 6th ed.; the diagram below is from the same source] Heat is lost from the lateral surface of a thin rod of length L into a surrounding medium that has a temperature of 0 K. If the linear law of heat transfer applies, then the heat equation has the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$



where h is a constant. Find an expression for the temperature $u(x,t)$ if the initial temperature is $f(x)$ over the interval $[0, L]$ and the ends $x = 0$ and $x = L$ are insulated as shown in the figure to the right. *Hint*: The insulation defines the boundary conditions.

2. [adapted from Prob. 15 in Sec. 13.4 of D. G. Zill, 6th ed.] A string is stretched and secured on the x-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the governing wave equation has the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t}, \qquad 0 < \beta < 1, \quad t > 0\,.$$

Find an expression for the displacement $u(x,t)$ if the string starts from rest from the initial displacement $f(x)$. A string that starts from rest has $g(x) = 0$.

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3. [adapted from Prob. 9 of Sec. 14.2 of Zill, 6th ed.] The temperature in a circular plate of radius c can be determined via the boundary value problem described below. The origin of the coordinate system, where $r = 0$, is at the center of the plate. Solve for $u(r, t)$, including finding the expressions needed to determine the coefficients in the series expansion.

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}, \qquad 0\le r\le c \quad \text{and} \quad t\ge 0$$

$$u(c,t)=0, \quad t\ge 0 \quad \text{and} \quad u(r,0)=f(r), \quad 0\le r\le c$$

4. Derive an explicit finite difference update equation for the numerical solution of the two-dimensional heat equation problem given by

$$c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}, \qquad 0\le x\le a, \quad 0\le y\le b, \quad \text{and} \quad t\ge 0,$$

where c is the thermal diffusivity of the material. The spatial locations within the solution space are defined by $x = (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$ and $y = (j - 1)\Delta x$ for $j = 1, 2, 3, \ldots, N_y$, where N_x and N_y are the total number of discrete points in the x and y directions, respectively. The time variable is indexed as $t = (n - 1)\Delta t$ for $n = 1, 2, 3, \ldots, N_t$, where N_t is the total number of time steps. Group like terms in the update equation so that the number of floating-point operations is minimized. Use the derivation at the beginning of Sec. 16.2 of the textbook (D. G. Zill, *Advanced Engineering Mathematics*, 6th ed.) as a guide, and use the following constants to simplify the update equation:

$$C_x=\frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_y=\frac{c\Delta t}{\Delta y^2}.$$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Find the forward, backward, and centered difference approximations to the first derivative of the functions shown below at the indicated points using the indicated intervals. Then find the centered difference approximation to the second derivative. Finally, find the percentage error of each derivative relative to the actual derivative obtained via analytical evaluation.

- **a.** $f(t)=2.4\cos(120\pi t)$ at $t = 4.0$ ms with $\Delta t = 1.0$ ms
- **b.** $f(x)=\sqrt{x-2}$ (principal value) at $x = 2.5$ m with $\Delta x = 10$ cm

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2. Recall that the homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \qquad 0\le x\le L \quad \text{and} \quad t\ge 0$$

$$\text{with } u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x),$$

where the dependent variable u represents the heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SoV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=-\lambda,$$

where $-\lambda$ is the separation constant. The resulting set of ordinary differential equations (ODEs) is

$$X''+\lambda X=0 \qquad \text{and} \qquad T'+k\lambda T=0,$$

and the resulting solution is

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t/L^2}\sin\left(\frac{n\pi}{L}x\right), \quad \text{where} \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Note that the thermal diffusivity k could have been associated with the ODE in $X(x)$ instead of the ODE in $T(t)$. That is, the application of the SoV method could have led to the alternate expression

$$k\frac{X''}{X}=\frac{T'}{T}=-\lambda.$$

Show that the solution for $u(x, t)$ given above is still obtained when the alternate expression is used as the starting point.

3. Find a SoV solution to the 1-D heat equation given below left, where u = temperature, for a rod of length $L = 2$ m and insulated ends. The solution must include the numerical values of the set of coefficients $\{A_n\}$, which has only two nonzero values. The initial temperature distribution $f(x)$ along the rod is given below. The thermal diffusivity of the rod is $k = 0.01\ \text{m}^2/\text{s}$. Use the hint below right to show that the coefficients for $n = 3$ and higher are all zero. You may use mathematical analysis software such as *Matlab* or *Mathematica* or a calculator to perform the numerical integrations required to obtain the two coefficient values.

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \qquad f(x)=200\sin^2\left(\frac{\pi x}{L}\right) \qquad \textit{Hint}\text{: } \sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$$

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4. [adapted from Prob. 18 of Sec. 14.2 of Zill, 6th ed.] The problem below describes the displacement u of a circular membrane with radius c that is vibrating in two dimensions. Complete the four parts listed below to find the SoV solution $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$.

$$a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial^2 u}{\partial t^2}, \qquad 0\le r\le c \quad \text{and} \quad t\ge 0$$

$$u(c,\theta,t)=0, \quad 0\le\theta\le 2\pi \quad \text{and} \quad t\ge 0$$

$$u(r,\theta,0)=f(r,\theta), \quad 0\le r\le c \quad \text{and} \quad 0\le\theta\le 2\pi$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(r,\theta), \qquad 0\le r\le c \quad \text{and} \quad 0\le\theta\le 2\pi .$$

a. Using the separation constants $-\lambda$ and $-\nu$, show that the separated ODEs are

$$T''+a^2\lambda T=0, \qquad \Theta''+\nu\Theta=0, \qquad \text{and} \qquad r^2R''+rR'+\left(\lambda r^2-\nu\right)R=0.$$

b. With $\lambda = \alpha^2$ and $\nu = \beta^2$, solve the separated equations in part **a**, and determine the eigenvalues and eigenfunctions of the problem.

c. Use the superposition principle to find a solution that involves a double sum. You do not have to find integral expressions for the coefficients.

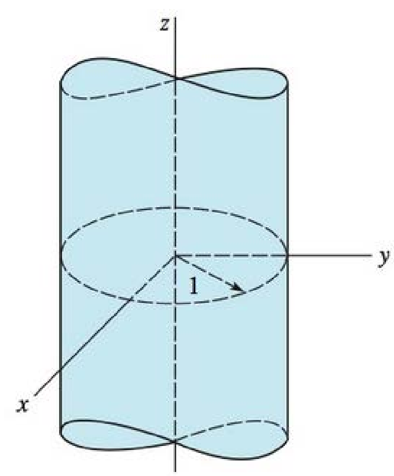
5. Find integral expressions for the four coefficients in the double sum in the previous problem, which considers a circular membrane that is vibrating in two dimensions.

6. [adapted from Prob. 11 of Sec. 14.2 of Zill, 6th ed.; the diagram below is from the same source] When there is heat transfer from the lateral side of an infinite circular cylinder of radius 1 (see figure below) into a surrounding medium at a temperature of 0 K, the temperature inside the cylinder is determined via the boundary value problem described below. The origin of the coordinate system, where $r = 0$, is at the center of the plate. Solve for $u(r, t)$, including finding the expression needed to determine the coefficients in the series expansion.

$$k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=\frac{\partial u}{\partial t}, \qquad 0\le r\le 1 \quad \text{and} \quad t\ge 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1}=-hu(1,t), \qquad h>0 \quad \text{and} \quad t\ge 0$$

$$u(r,0)=f(r), \quad 0\le r\le c$$



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7. D'Alembert's solution to the one-dimensional wave equation in unbounded media, defined by

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad -\infty \le x \le \infty \quad \text{and} \quad t \ge 0$$

$$u(x,0) = f(x) \quad \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = g(x)$$

is given by

$$u(x,t) = \frac{1}{2} f(x+at) + \frac{1}{2} f(x-at) + \frac{1}{2a} \int_{x-at}^{x+at} g(s)\, ds\ .$$

Consider a vibrating string problem in which the dependent variable u represents the displacement of the string from its equilibrium position, and a = 300 m/s, $f(x)$ = 0, and $g(x)$ is a rectangular pulse function given by

$$g(x) = \text{Rect}(x) = \begin{cases} A, & -0.25\ \text{m} \le x \le 0.25\ \text{m} \\ 0, & \text{elsewhere} \end{cases}$$

where A = 900 m/s. This simulates what would happen if someone were to strike the string with a 50 cm wide flat object. Plot the solution $u(x, t)$ over the spatial interval $x = -2$ m to x = 2 m at the times t = 0.5, 1, 2, and 4 ms. You will probably want to use mathematical analysis software like *Matlab* or *Mathematica* to produce the plots.

Hint: Consider evaluating the integral in the solution for six different cases (r = 0.25 m, the half-width of the object that strikes the string):

$x - at < -r$ and $x + at < -r$

$x - at < -r$ and $-r < (x + at) < r$

$x - at < -r$ and $x + at > r$

$-r < (x - at) < r$ and $-r < (x + at) < r$

$-r < (x - at) < r$ and $x + at > r$

$x - at > r$ and $x + at > r$