ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Homework Assignment #4 – due via Moodle at 11:59 pm on Thursday, Nov. 7, 2024

***Instructions, notes, and hints*:**

Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the posted solutions and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Submit only the graded problems by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

1. Find the eigenvalues and eigenfunctions of the boundary value problem (BVP) given below. Also specify the form of the inner product used to test the orthogonality of the solutions $\{y_n(x)\}$.

$$x^2 y'' + xy' + \lambda y = 0 \text{ with } \lambda = \alpha^2,\ \alpha > 0 \text{ and BCs } y(1) = 0 \text{ and } y(5) = 0$$

2. Show that the BVP in the previous problem has only the trivial solution if $\lambda < 0$.

3. Show that the solution to the boundary value problem given below with the indicated boundary conditions is the expression for $y(x)$ given farther below with the indicated eigenvalues.

$$y'' + a^2 y = 0,\ a > 0 \quad \text{with} \quad y(\pi/4) = 0 \text{ and } y(\pi) = 0$$

$$y(x) = C\left[-\tan(a_n\pi)\cos(a_n x) + \sin(a_n x)\right], \text{ where } C \text{ is a constant and } a_n = \frac{4}{3}n;\ n = 1, 2, 3, \ldots$$

4. Attempt to use the separation of variables method to decompose the following partial differential equations into a set of ordinary differential equations (ODEs). For the problem that involves more than two independent variables, you will need more than one separation constant. You do not have to find the solutions to the ODEs.

a. $\dfrac{\partial^2 E}{\partial x^2} + \dfrac{\partial^2 E}{\partial y^2} + \dfrac{\partial^2 E}{\partial z^2} = 0$ (Use separation constants k_x^2, k_y^2, and k_z^2)

b. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

c. $y\dfrac{\partial^2 u}{\partial x \partial y} + u = 0$

d. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}$, where k is a constant and $k > 0$

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***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Find solutions to the following boundary value problems (BVPs). Not all are eigenvalue problems (EVPs).

a. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$
b. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 1$
c. $x^2 y'' + xy' + a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$
d. $x^2 y'' + xy' - a^2 x^2 y = 0$ where $y(0)$ is finite and $y(1) = 0$

2. The norm and square norm of a function are defined by the expressions given below left. The square norm is also the self-product of a function. Find the square norm of the Fourier eigenfunction $y_n(x) = \cos(n\pi x)$, $n = 0, 1, 2, \ldots$ over the interval [0, 1] using analytical means. Provide a general result that applies for all values of n. You may look up the solution in an integral table to check your answer, but you must show the solution. *Hint*: Use a trigonometric identity or the identity given below right.

norm: $\|y_n\| = \sqrt{\int_a^b y_n^2(x)\,dx}$

square norm: $\|y_n\|^2 = \int_a^b y_n^2(x)\,dx$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

3. Chebyshev's equation is given below. For appropriate boundary conditions, it leads to an eigenvalue problem with orthogonal solutions $\{T_n(x)\}$ for $n = 0, 1, 2, \ldots$ that correspond to the eigenvalues $\{\lambda_n\}$. The solutions take the form of polynomials of order n; that is, as n increases, so does the order of the polynomial. Find the interval of x over which the solutions are orthogonal, and specify the form of the inner product used to test the orthogonality of the solutions $\{T_n(x)\}$.

$$(1-x^2)\,y'' - xy' + n^2 y = 0$$

4. The textbook (Zill, 6th ed.) presents a proof for the orthogonality of the nontrivial solutions to a Sturm-Liouville problem by starting with the self-adjoint form for two different eigenvalues and eigenfunctions given by

$$\frac{d}{dx}\left[r(x)\frac{dy_m}{dx}\right] + q(x)\,y_m + \lambda_m p(x)\,y_m = 0$$

$$\frac{d}{dx}\left[r(x)\frac{dy_n}{dx}\right] + q(x)\,y_n + \lambda_n p(x)\,y_n = 0$$

and then stating that multiplying the first equation by y_n and the second by y_m, subtracting the two equations, and finally integrating by parts from $x = a$ to $x = b$ yields the expression

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$$(\lambda_m - \lambda_n)\int_a^b p(x)\, y_m(x)\, y_n(x)\, dx = r(b)\left[y_m(b)\, y_n'(b) - y_n(b)\, y_m'(b)\right] \\ - r(a)\left[y_m(a)\, y_n'(a) - y_n(a)\, y_m'(a)\right].$$

Work through the missing steps to prove the orthogonality condition above.

5. Attempt to use the separation of variables method to find a solution to the following partial differential equation.

$$\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial u}{\partial x} = 0$$

6. Recall that the homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

$$\text{with } u(0,t) = 0 \quad u(L,t) = 0 \quad u(x,0) = f(x),$$

where the dependent variable u represents the relative heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SoV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = -\lambda,$$

where $-\lambda$ is the separation constant. The resulting set of ordinary differential equations (ODEs) is

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0,$$

and the resulting solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi}{L}x\right), \quad \text{where} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Note that the thermal diffusivity k could have been associated with the ODE in $X(x)$ instead of the ODE in $T(t)$. That is, the application of the SoV method could have led to the alternate expression

$$k\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

Show that the solution for $u(x, t)$ given above is still obtained when the alternate expression is used as the starting point.