ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Homework Assignment #3 – due via Moodle at 11:59 pm on Friday, Oct. 11, 2024

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the solutions when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

1. For each of the matrices below, determine how many of the singular values would be zero if the matrix were decomposed using the SVD method. Explain how you arrived at your answer. You may use *Matlab* or other software to check your answers, but your solution to each part must indicate that you understand how to predict the number of singular values.

a. $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 2 & 1 & 1 \\ -4 & -2 & -2 \\ 8 & 4 & 4 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

2. Starting with the result of the product AA^T, manually find expressions in terms of ε for the singular values and the orthogonal matrix U of the following matrix. The quantity ε is a small value ($\varepsilon << 1$). Find the singular values and U for the case when $\varepsilon = 0$, and briefly explain the implications of the results, especially how U changes (or not) for the cases when $\varepsilon = 0$ and $\varepsilon \neq 0$. You may check your answers using *Matlab*.

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}$$

3. Find solutions to the Fourier equation $y'' + a^2 y = 0$, with $a > 0$, for the following sets of boundary conditions. After you complete the solutions, briefly explain the implications of the results.

a. $y(0) = 1$ and $y(4) = 0$
b. $y(0) = 0$ and $y(4) = 0$

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4. **4.** Find solutions to the modified Fourier equation $y'' - a^2 y = 0$, with $a > 0$, for the following sets of boundary conditions. After you complete the solutions, briefly explain the implications of the results.

   a. $y(0) = 1$ and $y(4) = 0$
   b. $y(0) = 0$ and $y(4) = 0$

5. Find the solutions of the BVP given below. Consider the conditions $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

$$y'' + \lambda y = 0 \text{ with } y(0) = 0 \text{ and } y'(\pi/2) = 0$$

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

1. Show that an orthonormal matrix that multiplies a vector does not change the length of the vector. That is, show that $\mathbf{x}$ and $Q\mathbf{x}$ (where Q is an orthonormal matrix) have the same length. Note that the length of a vector is given by $x = (\mathbf{x}^T\mathbf{x})^{1/2}$.

2. Show that an orthonormal matrix does not change the angle between two vectors (i.e., it does not change their relative orientation). To do this, show that $\theta_{xy} = \theta_{ab}$, where $\mathbf{a} = Q\mathbf{x}$ and $\mathbf{b} = Q\mathbf{y}$. Note that the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by the expression

$$\cos\theta_{xy} = \frac{\mathbf{x}^T\mathbf{y}}{xy},$$

where x and y are the lengths of the vectors $\mathbf{x}$ and $\mathbf{y}$, respectively.

3. Starting with the result of the product A^TA, manually find the singular values of the following matrix. You may check your answer using *Matlab*.

$$\begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$$

4. Determine whether the following problems are initial value problems (IVPs), boundary value (BVPs), or neither. For both differential equations, $a > 0$. Briefly explain your answer for each part.

   a. $y'' + a^2 y = 0$ with $y(0) = 1$ and $y(1) = 0$
   b. $y'' + a^2 y = 0$ with $y(0) = 0$ and $y'(0) = 1$

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5. Find the solution to the BVP given below. Note that the boundary conditions are not homogeneous. Consider the conditions $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

$$y'' + \lambda y = 0 \text{ with } y(0) = 1 \text{ and } y'(\pi/2) = 0$$

6. Find the eigenvalues and eigenfunctions of the BVP given below. Consider the conditions $\lambda < -1$, $\lambda = -1$, and $\lambda > -1$.

$$y'' + (\lambda + 1)\, y = 0 \text{ with } y'(0) = 0 \text{ and } y'(1) = 0$$

7. [adapted from Prob. 30 in Sec. 3.9 of Zill, 6th ed.] The temperature distribution $u(r)$ in a circular ring or annulus is determined from the BVP given by

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0 \text{ with } u(a) = u_0 \text{ and } u(b) = u_1,$$

where u_0 and u_1 are constants. Show that

$$u(r) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}$$