# ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

**Homework Assignment #1 – due via Moodle at 11:59 pm on Friday, Sept. 13, 2024**
**[Prob. 5 revised 9/12/24]**

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

It is your responsibility to review the solutions when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above. Do not submit the ungraded problems.

***Graded Problems*:**

**1.** A *permutation matrix* is a square matrix consisting of only zeros and ones that can rearrange the rows of another matrix by pre-multiplying it or the columns by post-multiplying it.

- **a.** Verify that the matrix $P_{12}$ below interchanges rows 1 and 2 of the $3 \times 3$ matrix $A$.
- **b.** Find a permutation matrix $P_{13}$ that interchanges rows 1 and 3 of the $3 \times 3$ matrix $A$.

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

**2.** [adapted from Problem 12 from Zill, 6th ed., Sec. 8.7] In the system of equations shown below, the coefficient matrix $A$ in the matrix representation is almost singular if $\varepsilon$ is close to 1. Put another way, the row vectors in the matrix are almost linearly dependent if $\varepsilon \approx 1$. This type of system is called *ill-conditioned* because small changes in the coefficients lead to proportionately large changes in the solution; that is, the solution is very sensitive to the coefficient values. The elements of the solution vector also tend to be orders of magnitude larger than the coefficients and the elements of **b** (the right-hand-side vector). To illustrate these effects, find the solution of the system below for:

- **a.** $\varepsilon = 0.99$
- **b.** $\varepsilon = 1.01$
- **c.** $\varepsilon = 1.02$

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + \varepsilon x_2 &= 2 \end{aligned}$$

**3.** For the system of equations in the previous problem, show that the matrix is close to singular by finding its determinant for cases **a** through **c**. Also find the determinant of its inverse.

(*continued on next page*)

**4.** [adapted from Problem 17 from Zill, 6th ed., Sec. 8.3] Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the first, second, and third column vectors, respectively, of the matrix

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 1 & 0 & 2 \\ -1 & 5 & 13 \end{bmatrix}$$

Note that $2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. What is the maximum possible rank of $A$? Briefly explain your answer. *Hint*: Read the "Remarks" at the end of Sec. 8.3.

**5.** **[Text in boldface added 9/12/24]** An elementary row operation (ERO) that adds a multiple $\alpha$ of the $j^{\text{th}}$ row of an $N \times N$ matrix $A$ to its $k^{\text{th}}$ row can be expressed as a matrix multiplication $MA$, where the matrix $M$ is given by

$$M\left(j, k, \alpha\right) = I + \alpha \mathbf{e}_k \mathbf{e}_j^T,$$

where $\mathbf{e}_j$ is an $N \times 1$ column vector with a 1 in the $j^{\text{th}}$ entry and zeros everywhere else. Note that the $\mathbf{e}$ vectors are orthogonal (i.e., $\mathbf{e}_j^T\mathbf{e}_k = 0$) and that $k \neq j$ in the expression above (i.e., there is no reason to add a multiple of a row to itself).

a. Write out the full matrix for $M(2, 4, 3)$. **Assume that $N = 4$.**
b. Show that the inverse of $M(j, k, \alpha)$ is given by $M^{-1}\left(j, k, \alpha\right) = I - \alpha \mathbf{e}_k \mathbf{e}_j^T$.
c. Verify the expression for the inverse in part b by finding the inverse of $M(2, 4, 3)$ directly either by hand or by using your calculator or software. **Assume that $N = 4$.**
d. This type of ERO does not change the determinant of the matrix that it is applied to. That is, $|M(j, k, \alpha)\, A| = |A|$, so $|M(j, k, \alpha)| = 1$. Show that $|M(j, k, \alpha)| = 1$.

***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions (if applicable). Try not to give up too quickly if you struggle to solve any of them. Move on to a different problem and then come back to the difficult one after some time has passed.

**1.** [adapted from Problem 16 from Zill, 6th ed., Sec. 8.3] Let $A$ be a nonzero $4 \times 6$ matrix.

a. Find the maximum rank that $A$ can have.
b. Suppose that $\text{rank}(A|\mathbf{b}) = 2$. Find the values(s) of $\text{rank}(A)$ for which the system $A\boldsymbol{x} = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, is consistent and for which it is inconsistent.
c. Suppose that $\text{rank}(A) = 3$. Find the number of parameters that the solution of the system $A\boldsymbol{x} = \mathbf{0}$ must have.

**2.** Use Example 10 in Sec. 8.2 of Zill, 6th ed. as a guide to balance the chemical equation $Ca_3(PO_4)_2 + H_3PO_4 \rightarrow Ca(H_2PO_4)_2$.

Try working through all or most of Problems 35 through 44 at the end of Sec. 8.6 of Zill, 6th ed. They are all "show that…" problems, so you should be able to verify on your own whether you have solved them correctly.