ENGR 695 Advanced Topics in Engineering Mathematics Fall 2024

Final Exam General Information

The final exam will take place **11:45 am–2:45 pm on Monday, December 16 in Breakiron 065**. The exam will be designed to be completed in less than 1.5 hours, but the exam period will be three hours long. **You must begin the exam at 11:45 am.** You will not be allowed to take the exam if you arrive after the first student has completed it and left the room.

The first page of the exam will include the following instructions, although some might be shortened or paraphrased:

The following policies will be in effect for the exam:

1. You will be allowed to use a non-wireless enabled calculator, such as a TI-99. You must put away smart watches, cell phones, tablets, laptops, and other devices that can connect to the internet wirelessly. I will record your calculator make and model during the exam.
2. You will be allowed to use an unlimited number of paper help sheets. These could include the notes that you recorded in class (either loose or collected in a notebook), homework solutions, supplemental notes from the course Moodle site, copied or printed pages from the textbook, or any other printed or handwritten material. Hard copies of the textbook will not be allowed since not every student has access to one. A copy of the linear system solvability flowchart (Fig. 8.3.1 of Zill, 6th ed.) will be provided. If there is anything else that you feel would be helpful, please notify me.
3. If you begin the exam after the start time, you must complete it in the remaining allotted time. However, you will not be allowed to take the exam if you arrive after the first student has completed it and left the room. The latter case is equivalent to missing the exam.
4. **You may not leave the exam room without prior permission except in an emergency or for an urgent medical condition. Please use the restroom before the exam.**

Solutions to the final exam will not be posted, but you may review your final exam and discuss it with me at any time after it has been graded, even next year. Your final exam score will be posted at the course Moodle site and will remain there indefinitely.

A list of topics to be covered on the exam begins on the next page.

Review Topics for ENGR 695 Final Exam

Below is a list of topics that could appear in one form or another on the exam. Not all of these topics will be covered, and it is possible that an exam problem could cover a detail not specifically listed here. However, this list has been made as comprehensive as possible. You should be familiar with the topics on the previous review sheet in addition to those listed below.

The final exam is not cumulative in the sense that you will not be presented with problems whose solutions relate specifically to the material covered before the midterm exam. However, much of the material that was covered since then builds on the earlier material. (For example, the solution of a PDE using the separation of variables method could rely on the ability to find solutions to the Fourier or modified Fourier equations.) You will be expected to have a solid grasp of foundational material that is relevant to the second half of the semester.

Although significant effort has been made to ensure that there are no errors in this review sheet, some might nevertheless appear. The textbook and the supplemental readings are the final authority in all factual matters, unless errors have been specifically identified there. You are ultimately responsible for obtaining accurate and authoritative information when preparing for the exam.

Power series solutions to ODEs

- constant coefficients → find roots of characteristic equation; power series not necessary
- variable coefficients → power series solution

  $y(x)=\sum_{n=0}^{\infty} c_n x^n$
- power series approach works in general for any ODE but is unnecessary for constant coefficients and many forms of the Cauchy-Euler equation
- ordinary points: values of x at which variable coefficients are analytic
- singular points: values of x that are not ordinary points
  example: $\ln(x)$ is not analytic for $x \le 0$, so the ordinary points are $x > 0$
- power series solutions valid only over intervals for which they converge; often not valid over all of $-\infty < x < \infty$

Frequently encountered ODEs and their suggested solutions (there are many other ODEs besides these); most solutions are recommended for closed boundaries

- Fourier equation: $y'' + a^2 y = 0$

  solution: $y(x) = c_1 \cos(ax) + c_2 \sin(ax)$
- modified Fourier equation: $y'' - a^2 y = 0$

  solution: $y(x) = c_1 \cosh(ax) + c_2 \sinh(ax)$
- 2nd order Cauchy-Euler equation: $ax^2 y'' + bxy' + cy = 0$
  - try solutions of the form $y = x^m$
  - find roots of auxiliary eqn $am(m-1) + bm + c = 0 \quad \rightarrow \quad am^2 + (b-a)m + c = 0$
  - for distinct real roots m_1 and m_2, $y(x) = c_1 x^{m_1} + c_2 x^{m_2}$
  - for repeated real roots m_1, $y(x) = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

  - for complex conjugate roots $m = \alpha \pm i\beta$,
    $y(x) = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$
  - special case: $x^2 y'' + xy' - \alpha^2 y = 0$
    solutions are $y = \begin{cases} c_1 + c_2 \ln x, & \alpha = 0 \\ c_1 x^{-\alpha} + c_2 x^{\alpha}, & \alpha > 0 \end{cases}$
  - further special case: $x^2 y'' + xy' - y = 0$ ($\alpha = 1$)
    solution is $y = \frac{c_1}{x} + c_2 x$

- Bessel equation: $x^2 y'' + xy' + \left(x^2 - \nu^2\right) y = 0$ (ν often replaced with n)
  solution: $y(x) = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x)$ (Bessel functions of the 1st & 2nd kind)
- modified Bessel equation: $x^2 y'' + xy' - \left(x^2 + \nu^2\right) y = 0$ (ν often replaced with n)
  solution: $y(x) = c_1 I_{\nu}(x) + c_2 K_{\nu}(x)$ (modified Bessel functions of the 1st & 2nd kind)
- parametric Bessel equation: $x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0$
  solution: $y(x) = c_1 J_{\nu}(\alpha x) + c_2 Y_{\nu}(\alpha x)$
- modified parametric Bessel equation: $x^2 y'' + xy' - \left(\alpha^2 x^2 + \nu^2\right) y = 0$
  solution: $y(x) = c_1 I_{\nu}(\alpha x) + c_2 K_{\nu}(\alpha x)$

Inner product of two functions
- analogous to vector inner product
- definition:
  given two functions $y_m(x)$ and $y_n(x)$ and a weighting function $p(x)$,
  $\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x)\, y_n(x) p(x)\, dx$,
  where a and b specify the interval over which the inner product is defined
- norm: $\|y_n\| = \sqrt{\int_a^b y_n^2(x)\, p(x)\, dx}$; square norm (self-product): $\|y_n\|^2 = \int_a^b y_n^2(x)\, p(x)\, dx$

Boundary conditions
- Dirichlet: conditions on dependent variable [e.g., $y(L) = 0$]
- Neumann: conditions on derivative(s) of dependent variable [e.g., $y'(L) = 0$]
- Robin: conditions on dependent variable and derivative(s) combined
  [e.g., $A_1\, y(L) + B_1\, y'(L) = C_1$]
- definition of homogeneous BVP: right-hand side of ODE *and* BCs = 0

Regular Sturm-Liouville problem

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + q(x)\, y + \lambda\, p(x)\, y = 0 \quad \text{(self-adjoint form)}$$

- some textbooks use $p(x)$ in place of $r(x)$ and $w(x)$ in place of $p(x)$
- second-order with variable coefficients
- self-adjoint form allows identification of weight function $p(x)$ used in inner product
- λ is a parameter in the problem (eigenvalue)
- must have $r(x)$, $p(x) > 0$ over interval of solution (a, b) for a *regular* S-L problem
- boundary conditions defined over interval (a, b):
  $A_1 y(a) + B_1 y'(a) = 0$
  $A_2 y(b) + B_2 y'(b) = 0$

Conversion of second-order ODE to self-adjoint Sturm-Liouville form (not always possible):

$$a(x)y'' + b(x)y' + c(x)y + \lambda d(x)y = 0 \quad \rightarrow \quad \frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + q(x)y + \lambda p(x)y = 0$$

a. compute the integrating factor $\mu(x)$ (watch out for $a(x) = 0$ for any x over the bounded interval):

$$\mu(x) = \exp\left(\int \frac{b(x)}{a(x)}dx\right)$$

b. compute the elements of the S-L self-adjoint form:

$$r(x) = \mu(x) \qquad q(x) = \frac{c(x)}{a(x)}\mu(x) \qquad p(x) = \frac{d(x)}{a(x)}\mu(x)$$

c. verify that $r(x)$, $p(x) > 0$ over interval of solution for regular S-L problem

Orthogonality conditions on solutions to Sturm-Liouville problem

- relationship between inner product and boundary conditions (BCs)

$$(\lambda_m - \lambda_n)\int_a^b p(x)y_m(x)y_n(x)dx = r(b)\left[y_m(b)y_n'(b) - y_n(b)y_m'(b)\right] - r(a)\left[y_m(a)y_n'(a) - y_n(a)y_m'(a)\right]$$

- implications:
  - BCs must be homogenous if the solutions y_m and y_n are to be orthogonal. If $m \neq n$ and the BCs are homogeneous, then the right-hand side equals zero
  - y_m and y_n can be orthogonal if $r(x) = 0$ at one of the boundaries and the BC at the other boundary is homogeneous or if $r(a) = r(b) = 0$

Singular Sturm-Liouville problem

- addresses case when $r(x) > 0$ is not satisfied at one or both boundaries
- right-hand side of equation in inner product relationship above is zero when:
  - $r(a) = 0$ and $y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0$
  - $r(b) = 0$ and $y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0$
  - $r(a) = r(b) = 0$ and no BCs are specified at $x = a$ or $x = b$
  - $r(a) = r(b)$ and the BCs are $y(a) = y(b)$ and $y'(a) = y'(b)$ (periodic BCs)
  - the solutions $\{y_n\}$ are orthogonal if $r(a) = 0$ and/or $r(b) = 0$ provided that the solutions are bounded (i.e., do not approach ∞) at the corresponding boundary

The "Sturm-Liouville Insurance Policy" (SLIP)

a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
b. There is an infinity of eigenvalues.
c. There is a smallest but not a largest eigenvalue.
d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ such that $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$).
e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$. That is, their inner product is

$$\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x)y_n(x)p(x)dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

where C_m is some non-zero constant

g. The set of solutions is complete; it forms a basis for the space of square-integrable functions on the interval [a, b]:

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x), \quad \text{where} \quad a_n = \frac{\langle f(x), y_n(x) \rangle}{\langle y_n(x), y_n(x) \rangle} = \frac{\langle f(x), y_n(x) \rangle}{\|y_n(x)\|}$$

Solution of partial differential equations (PDEs) using the separation of variables (SoV) method

- not always possible to find a solution with this method; some PDE solutions are not separable
- overall solution assumed to be product of solutions in each independent variable
- solution to ODE in one independent variable could be an infinite sum of eigenfunctions corresponding to different eigenvalues (superposition principle)
- focus on three commonly encountered homogeneous equations (variations are possible):
  - heat equation: $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k \geq 0$
  - wave equation: $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$
  - Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- boundary conditions:
  - Dirichlet – apply to dependent variable at boundaries
  - Neumann – apply to spatial derivative of dependent variable at boundaries
  - Robin – relationship b/w dependent variable and its spatial derivative
  - open – no physical boundaries; solution space extends indefinitely
- typical initial conditions:
  - $u(x, 0) = f(x)$
  - $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$
- separated 2nd-order ODEs could have eigenvalues that satisfy $\lambda < 0$, $\lambda = 0$, and/or $\lambda > 0$; form of solution depends on which one(s) lead(s) to non-trivial solutions
- solution usually takes form of infinite series of eigenfunctions in all independent variables
- coefficients in infinite series are determined using inner products of orthogonal eigenfunctions
- interpretation of solutions
  - helpful to determine whether solution is correct (does it make sense?)
  - heat/diffusion problems – does time progression of solution correspond to heat/concentration gradients?
  - wave equation – solutions are typically standing waves at harmonically related frequencies (in Cartesian coordinate system) if solution space is bounded; equivalent to superpositions of counterpropagating traveling waves
  - Laplace's equation – solution must satisfy the maximum principle (max and min values on boundary; no local max/min extrema in interior of solution space)
- d'Alembert's solution for wave equation with open boundary conditions

Problems in cylindrical coordinates

- after separation, one of the ODEs is typically in the form of the parametric Bessel equation ($r^2R'' + rR' + \lambda r^2 R = 0$) or Cauchy-Euler equation ($r^2R'' + rR' - \lambda R = 0$)
- requirement that solution be finite at $r = 0$ usually eliminates $Y_n(\alpha x)$ or r^{-n} as a solution
- usually necessary to find the zeroes of Bessel function; can be derived using special numerical methods or can look up in widely available tables
- periodicity condition might be applied to one of the ODEs; e.g., $\Theta(\theta) = \Theta(\theta + 2\pi)$

Numerical solutions of PDEs

- replace derivatives in PDEs with finite difference approximations
- types of finite differences (can be applied in time or spatial domain), where Δx is a spatial interval of small but non-zero or non-infinitesimal size
  - forward difference: $\frac{\partial u(x)}{\partial x} \approx \frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}$
  - backward difference: $\frac{\partial u(x)}{\partial x} \approx \frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}$
  - centered difference: $\frac{\partial u(x)}{\partial x} \approx \frac{u(x+\Delta x,t)-u(x-\Delta x,t)}{2\Delta x}$
- centered second-order finite difference: $\frac{\partial^2 u(x)}{\partial x^2} \approx \frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2}$
- for time-domain finite differences, Δt is used, where Δt is a time interval of short but non-zero duration
- shorthand notation (n might be used in place of j for the time index number)
  - $u_{i,j} = u(x,t)$
  - $u_{i+1,j} = u(x+\Delta x,t)$
  - $u_{i-1,j} = u(x-\Delta x,t)$
  - $u_{i,j+1} = u(x,t+\Delta t)$
  - $u_{i,j-1} = u(x,t-\Delta t)$
- explicit vs. implicit solutions
- explicit finite difference solutions usually have stability criterion that limits value of Δt
  - for explicit FD solution of 1-D heat equation (must modify for 2-D or 3-D), $\frac{c\Delta t}{\Delta x^2} \le 0.5$, where c = thermal diffusivity
  - for explicit FD solution of 1-D wave equation (the Courant-Friedrichs-Lewy, or CFL, condition; must modify for 2-D or 3-D), $\frac{v_p \Delta t}{\Delta x} \le 1$, where v_p = propagation velocity
- Crank-Nicholson algorithm
  - applies specifically to the heat/diffusion equation
  - implicit, which requires solution of a system of equations (matrix equation) at every time step
  - inherently stable; no restriction on time step size for stability
  - can be expressed in matrix form, which might enhance efficiency
- incorporation of boundary conditions into FD solutions
  - Dirichlet – simply set solution values at boundaries equal to the specified BCs
  - Neumann – apply FD approximation (centered type, if possible) and combine with update equation that applies to interior points in the solution space
  - Robin – similar to Neumann
  - open – for wave equation, use FD approximation of one-way wave equation and combine with update equation that applies to interior points in the solution space (more sophisticated open BCs are available as well)

- accuracy considerations
  - accuracy generally improves as spatial step size Δx and/or time step size Δt is decreased, although not always (e.g., CFL condition is most accurate setting for Δt in explicit FD solution of wave equation)
  - grid dispersion: artificial change in velocity (usually frequency dependent) due to discretization in space; can reduce by using small spatial step sizes
  - grid dissipation: artificial attenuation (usually frequency dependent) due to discretization in space; can reduce by using small spatial step sizes
  - finite precision of computer representation of numbers becomes a problem for very small step sizes
  - decreasing Δx and/or Δt leads to increased use of computational resources (e.g., memory & execution time)
- FD solutions in non-Cartesian coordinate systems can be challenging due to more complicated expressions and variable step sizes. It is sometimes better to use the Cartesian coordinate system and apply a staircase approximation for curving or otherwise irregular boundaries or interfaces between different materials.
- If more than one type of material is present in the solution space, then additional interior boundary conditions might have to be imposed. Some FD methods can account for interior boundaries simply and directly.

Relevant course material:

HW: #4 through #7 (#7 is ungraded)
Labs: #6 through #10
Reading: Assignments from Oct. 7 through Dec. 9, including the supplemental readings:
"Series Solution Method for ODEs"
"Meet the Bessels"
"Sturm-Liouville Problems: Eigenfunction Orthogonality"
"D'Alembert's Solution of the 1-D Wave Equation Problem in Unbounded Media"
"Neumann Boundary Conditions in the Explicit Finite Difference Solution of the 1-D Heat Equation"
"Open Boundaries in the Finite Difference Solution of the 1-D Wave Equation"