Lesson 9: Summary

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1 Learning goals for this lesson

The following is a list of basic ideas that were covered during this lesson. You must be fairly conversant with all of them now. If you have any questions, please post them on Piazza. The link is `piazza.com/phys211`.

- Moment of inertia of 2D objects - how to calculate
- Rolling - translation and rotation, the no-slip condition.

2 A quick summary

This section does not aim at being comprehensive. It's more of a slightly expanded version of the above, and primary serves to jog your memory, and be a quick look back place in case you forget something. If any of this is unclear, please post a question!

In this lesson, we will continue from lesson 8. We learn to compute the moment of inertia of two dimensional objects. As an example we will calculate the moment of inertia of a disc abouta perpendicular axis going through it's center. Chaning the axis of rotation will change the moment of inertia since the mass distribution will be different around the new axis.

Shown in fig. 1 is a disc with a perpendicular axis, and its top view. Like before, we calculate the moment of inertia by dividing the disc into several tiny pieces, and adding their moments of inertia to obtain that of the disc. Since a disc is circular, it is natural to describe a tiny piece by its length along the radius and along the arc. We see one such piece in the figure. Let's assume the mass of this piece is dm. The density of the disc is the mass over area, i.e., $M/\pi R^2$. The area of the tiny piece dm is given by (treating it as a tiny rectangle),

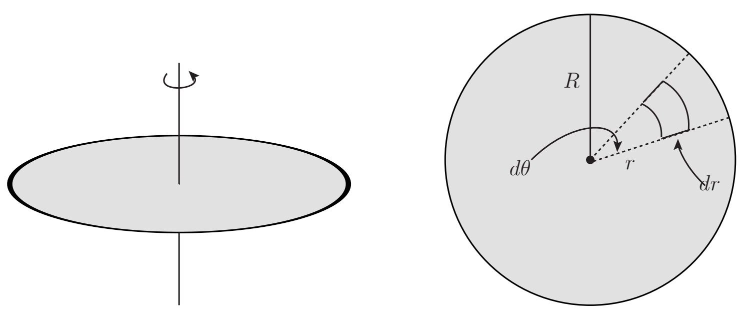

Figure 1: Disc with an axis perpendicular to it through it's center.

$$dA = r d\theta\, dr \tag{1}$$

The mass of this piece is therefore,

$$dm = \rho dA = \frac{M}{\pi R^2} r dr\, d\theta \tag{2}$$

The moment of inertia of this tiny piece is,

$$dI = dm\, r^2 = \frac{M}{\pi R^2} r^3\, dr\, d\theta. \tag{3}$$

In order to find the moment of inertia of the entire disc, we sum over the entire disc, i.e., we integrate from $r = 0 \to R$ and $\theta = 0 \to 2\pi$. This gives us,

$$\begin{aligned} I &= \int_{r=0}^{R} \int_{\theta=0}^{2\pi} \frac{M}{\pi R^2} r^3\, dr\, d\theta \\ &= \frac{M}{\pi R^2} \int_{r=0}^{R} r^3 dr \int_{\theta=0}^{2\pi} d\theta \\ &= 2\pi \frac{M}{\pi R^2} \int_{r=0}^{R} r^3 dr \\ &= 2\pi \frac{M}{\pi R^2} \frac{R^4}{4} \\ &= \frac{MR^2}{2}. \end{aligned} \tag{4}$$

So, the moment of inertia of a disc of mass M and radius R of uniform density is one half of that of a point having mass M sitting at radius R.

We will now go on to study rolling. Rolling, as we all know, is when an object both rotates and translates (i.e. moves along a line). We will study the motion of the center of mass, and the rotation about this. One could equivalently study any other points, but we're making the simplest choice. We'll first discuss rolling without slipping.

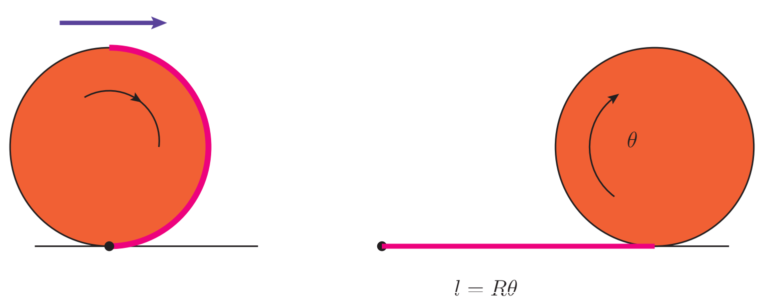

Figure 2: Rolling without slipping.

When an object rolls without slipping, it essentially means that the contact point at the bottom is not moving with respect to the ground (slipping) at any given instant. So the object keeps making contact at a new point as it rolls. We can translate this condition into an equation. Shown in fig. 2 is an circular wheel that rolls without slipping. If it moves forward a distance x, and rotates by an angle θ, then note that the arc length that it rotates through is precisely the distance travelled. This can be visualizes by thinking about a wheel painting a stripe on the road as it goes. The painted length of the stripe is precisely the arc length of the paint when it was on the wheel. That means,

$$x = R\theta \tag{5}$$

where R is the radius of the wheel. We can differentiate the above equation with respect to time once and twice, to get,

$$v = R\omega, \qquad a = R\alpha. \tag{6}$$

The rotation variables are therefore directly related to the translation variables.

The expression $v = R\omega$ is familiar from before. It is the tangential speed of a point at distance R from the center of an object rotating with angular speed ω. Well, this is precisely what's happening here, but since the road doesn't slip w.r.t to the wheel, the wheel moves forward with the same velocity!

If there is slipping when rolling occurs, then the angular and linear variables are not related and we have to calculate them separately.

3 Additional resources

- Look at the files area on angel for links.