Lesson 6: Summary

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1 Learning goals for this lesson

The following is a list of basic ideas that were covered during this lesson. You must be fairly conversant with all of them now. If you have any questions, please post them on Piazza. The link is `piazza.com/phys211`.

- The idea of centripetal acceleration as a force directed towards the center required for circular motion;
- How do we describe motion in a circular path - angles, angular velocity, etc.;
- Putting together friction and circular motion - examples of cars, banked roads.

2 A quick summary

This section does not aim at being comprehensive. It's more of a slightly expanded version of the above, and primary serves to jog your memory, and be a quick look back place in case you forget something. If any of this is unclear, please post a question!

We started the class with an activity where you were required to use a mallet to try and move a bowling ball in a circle. While it was tricky to get this working, we realized that in order to get it going in a circle, we need to keep hitting it "inwards" or towards the center of the circle. This in general is true. In circular motion, even if the speed of the object stays constant, the direction keeps changing, therefore producing a change in the velocity (remember velocity is a vector and has direction). The change is always directed towards the center of the circle as can be seen by drawing two adjacent velocity vectors and looking at the difference geometrically as shown in fig. 1

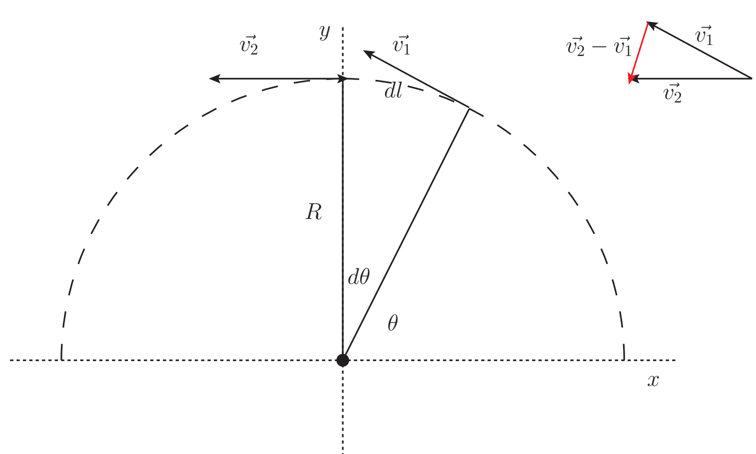

Figure 1: Difference in nearby velocities for circular motion.

We will later use this picture to calculate the magnitude of this acceleration. Before we get there, however, we need to learn how to describe motion in a circular path. We will only be concerned with particle moving in a plane, i.e., 2D circular motion. Figure 2 shows an object moving in a circle. A circle is defined by a

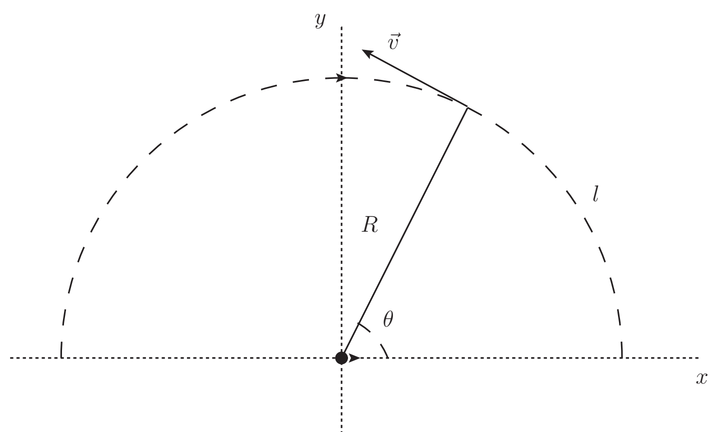

Figure 2: Circular motion.

radius, and nothing else. A collection of dots a fixed distance away (the "radius") from a point (the "center") is a circle. When an object moves in a circle, it's velocity is always directed along the tangent to the circle. So is it's displacement. If the particle speeds up or slows down, then this tangential velocity changes with time, producing a tangential acceleration. Note, that this is different from the centripetal acceleration that has to do with the direction of the tangential velocity changing. It is directed radially towards the center. As shown in fig. 2, and as we studied earlier for 2D motion, we can locate the particle by supplying it's x and y coordinates. As the particle goes around the circle, the x and y coordinates both change, and in fact oscillate back and forth. We could continue to obtain v_x, v_y and so on. However, there is an equivalent description that can be used. In a circle, we can locate a point by simply specifying at what angle it lies (angles being conventionally measure from the x-axis counterclockwise). Since the radius of the circle is know, we know exactly where this point is. The tangential velocity is then, the rate at which this point moves around in a circle, in other words, the rate at which the angle increases. If we denote the angle by θ, then the rate at which the angle increases with time, is given by

$$\omega \equiv \frac{d\theta}{dt} \tag{1}$$

and ω is called the angular velocity for obvious reasons. Note that since θ or angle is measured in radians, angular velocity is measured in rad/s. There is no length associated with angular velocity. If the tangential velocity changes w.r.t time, then we understand that the particle either speeds up or slows down in the tangential direction. We can define this change by

$$\alpha = \frac{d\omega}{dt} \tag{2}$$

and α is called the angular acceleration with the units rad/s^2. So, for uniform circular motion, i.e., with constant speed or tangential velocity, $\alpha = 0$.

An important relation to remember about a circle is that the arc length Δl enclosed by an angle $\Delta\theta$ is $R\Delta\theta$, R being the radius, and $\Delta\theta$ being measure in radians. It's a simple geometric fact about circles. The tangential velocity is then the rate at which an arc length is covered per unit time, giving us

$$v_t = \frac{dl}{dt} = \frac{d(R\theta)}{dt} = R\frac{d\theta}{dt} = R\omega. \tag{3}$$

Similarly, $a_t = R\alpha$.

We can now proceed to calculate the centripetal acceleration a_c that makes the particle go in a circle. Remember, this is due to the change in the **direction** of the tangential velocity as we go around the circle. Figure 1 shows a particle in circular motion, having traversed a small angle $\Delta\theta$ in a small time Δt. During this motion, the tangential velocity changes direction. The figure shows that the difference $\vec{v_2} - \vec{v_1}$ is given by a vector pointing towards the center of the circle (see fig. 1) with magnitude $v_t\Delta\theta$, v_t being the constant tangential velocity, i.e., $|\vec{v_1}| = |\vec{v_2}| = v_t$. The centripetal acceleration is this difference in velocity per unit time, i.e.

$$a_c = \frac{|\vec{v_2} - \vec{v_1}|}{\Delta t} = v_t \frac{\Delta\theta}{\Delta t} = v_t\omega = \frac{v_t^2}{R} = R\omega^2 \tag{4}$$

and points to the center of the circle. We've thus calculated the centripetal acceleration then every object in circular motion must undergo.

Note that we have as yet said nothing about what provides this acceleration. It of course is different in different cases. In the case of the ball and mallet, you striking the ball repeatedly provides a centripetal acceleration (although it's not really constant in this case). In the case of a ball being swung around on a string, the tension in the string provides the centripetal acceleration. We'll see three examples below.

Example 1 - ball on a string

A ball of mass m is swung in a horizontal circle or radius R and a speed v on a massless string. What is the tension in the string? A free body diagram is shown in fig. 3 Tension is the only force keeping the ball

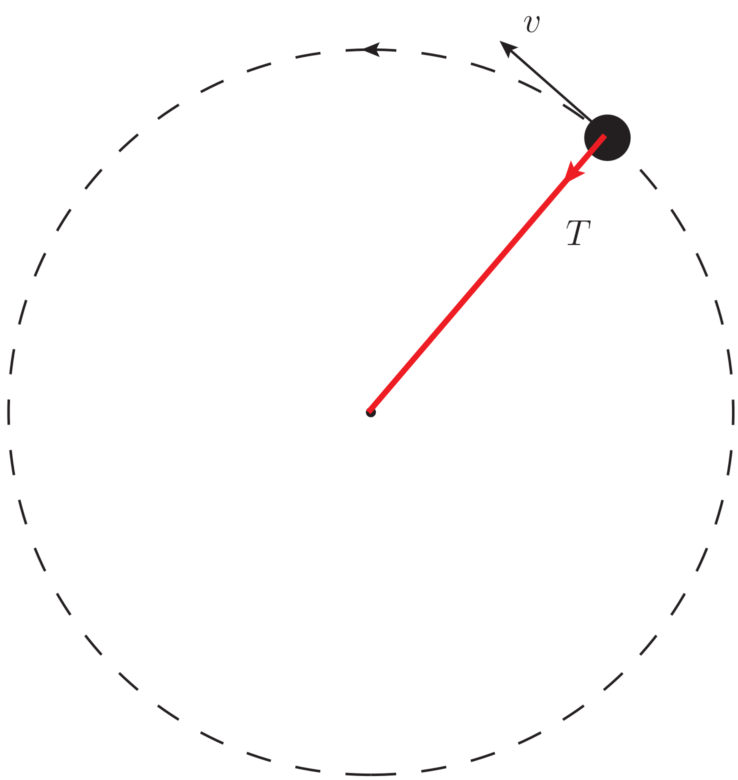

Figure 3: Ball on string

going in a circle, so Newton's law for the ball is

$$T = ma_c = \frac{mv^2}{R}. \tag{5}$$

So, that's the tension. A ball of mass 0.1kg going in a circle of radius 0.5m with a speed of 0.5m/s for example produces a string tension of $T = 0.1 \times 0.5^2/0.5 = 0.05N$.

Example 2 - car going in a circle of a horizontal road

A car of mass m is going around in a circle of radius $R = 7m$ as shown in figure 4. If the friction co-efficient between the tires and the road is $\mu = 0.6$, how fast can the car go without slipping out of the curve.

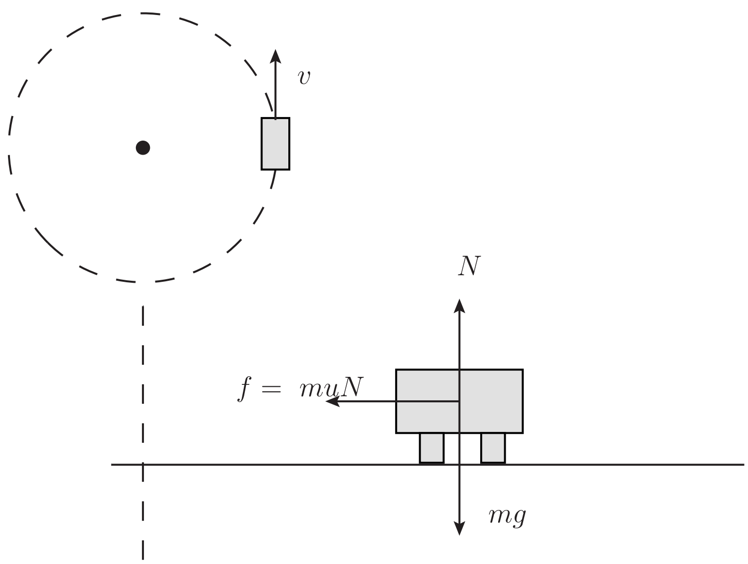

Figure 4: Car going in a circle of a horizontal road.

In this question, it is important to recognize that friction is the only force preventing the car from slipping out. So, all the centripetal acceleration is produced just by friction. It is therefore obvious that

$$F_{\text{max. friction}} = \mu N = \mu mg = ma_c = \frac{mv^2}{R} \tag{6}$$

giving

$$v = \sqrt{\mu g R}. \tag{7}$$

For the given values, we get $v \approx 6.5$m/s ≈ 15mph. Any faster, and friction wouldn't be able to provide the necessary centripetal acceleration since it is already at its highest value and the car would slip out. As we lower the friction coefficient the max. speed decreases, and if the road was frictionless, we get $v = 0$, which means you can't really turn!

Example 3 - car going in a circle of a banked road

To aid faster turning on curves, roads are usually banked. Let's see how this actually helps. Figure 5 shows the front view of a car on a banked road and the forces. Note that the normal force of the road on the car now is at an angle and has a component in the horizontal direction pointing towards the center of the curve. Friction also exists and it too has a component towards the center and one vertically upwards. Some of the

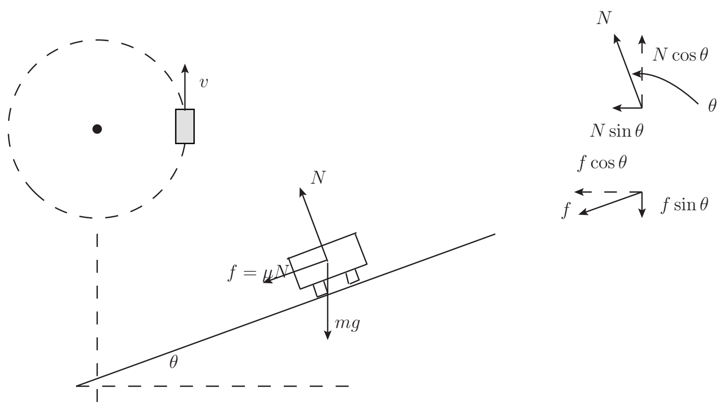

Figure 5: Car on banked road

centripetal acceleration can now be potentially provided by the normal force. Let's see how this works. In the vertical direction there is no motion and we get

$$N\cos\theta - mg - \mu N\sin\theta = 0 \implies N = \frac{mg}{\cos\theta - \mu\sin\theta}. \tag{8}$$

In the horizontal direction, we get,

$$N\sin\theta + \mu N\cos\theta = ma_c = \frac{mv^2}{R}. \tag{9}$$

Pluggin in N from the previous equation into the above equation and solving for v, we get,

$$v = \sqrt{gR\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}}. \tag{10}$$

Let's analyze this a bit. If the angle $\theta = 0$, then we should get the result of example 2. Putting $\theta = 0$ in the above equation, we get $v = \sqrt{\mu g R}$, which is indeed what we expect. Note one interesting thing now - even if $\mu = 0$, i.e. we're driving on ice, we can make turns at or slower than $v = \sqrt{gR\tan\theta}$. This is basically why roads are banked. You don't rely on friction so much any more. However, having friction helps and you corner faster!