Lesson 20 and 21: Gravitation

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1 Learning goals for this lesson

The following is a list of basic ideas that were covered during this lesson. You must be fairly conversant with all of them now. If you have any questions, please post them on Piazza. The link is `piazza.com/phys211`.

- Keplers laws and their meaning
- The inverse square law and Newtonian gravitation
- Gravitational potential energy

2 A quick summary

About 500 years ago, several scientists and astronomers of the time charted out the skies in a very detailed fashion. Data was recorded on transit times of planets, comets, etc. In the same century, an astronomer by the name of Kepler who worked with one of the very prolific astronomers of the time, Brahe, started observing patterns in some of the recorded data. He noticed that he could come up with empirical relations that described the motion of planets. The heliocentric view (sun at the center) of the solar system was already well understood by then. Kepler figured out that there were intimate relations between things like the time periods of the orbits of planets and their average distances from the sun, and other such relations. These have now come to be known as the "Kepler's laws" of planetary motion. There are essentially three of them. One states that planets orbit around the sun in elliptical orbits with the sun at one of the focii of the ellipse, as shown in 1. Further, he said that the area swept out by the radius vector (between the sun and the planet) per unit time was a constant for a given planet. And finally, that the time period of a planet (to make one revolution around the sun) obeyed $T^2 \propto r^3$, where r is the "semi-major axis" of the ellipse. In the case of a circular orbit, it is just the radius of the circle. Figure 1 shows the various geometric quantities.

Many attempts were made to explain these laws from a more fundamental standpoint - Hooke was among others to suggest that perhaps there was a force between any two massive objects that varied as the square of the inverse distance between the objects. The law, which finally was shown to predict Keplers laws and elliptical orbits by Newton, came to be known as Newtons law of gravitation. It states that any two (point) masses m_1 and m_2 attract each other by a force given by,

$$F = \frac{Gm_1 m_2}{r^2} \tag{1}$$

where r is the distance between them. m_1 and m_2 are "gravitational masses", and miraculously coincide with the "inertial mass" that shows up in $F = ma$. The inertial mass is a measure of how difficult it is for a given force to make an object accelerate, while the gravitational mass has to do with the gravitational force between these masses. There is no obvious reason as to why these masses should be the same physical quantity, but very accurate experiments (by Eötvos e.g.) have shown this equivalence to be true. The constant G is a numerical constant, that fixes the units and provides a scale for the gravitational force. It is called the constant of gravitation, and it is a constant of nature, like the speed of light.

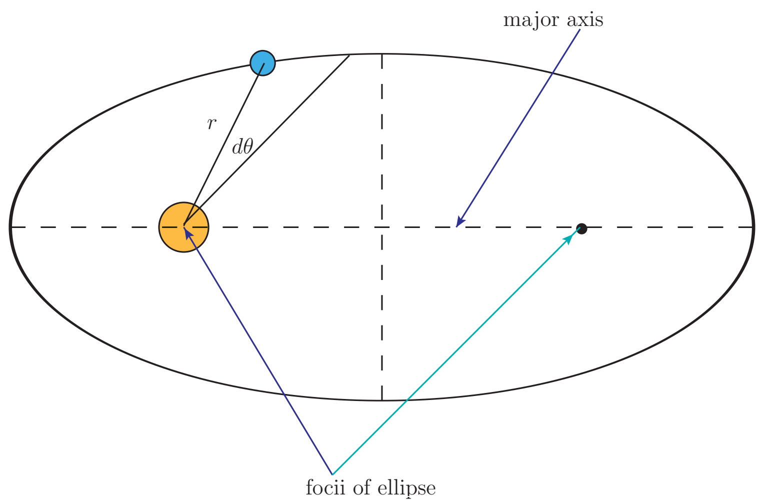

Figure 1: Elliptical orbit.

One can show that under the influence of such a force, an object can execute orbital motion in an elliptical orbit. More complex motion is also allowed. We will only concern ourselves with circular orbits. Most planets in our solar system have almost circular orbits, the orbits of satellites around the earth are almost circular, so from a practical consideration, this is sufficient for now.

Consider a satellite orbiting the earth at some height h above the earth. If the mass of the satellite is m, then the gravitational force acting on it is

$$F = \frac{GM_e m}{(R_e + h)^2}. \tag{2}$$

If it travels in a circular orbit, then its tangential velocity is constant. The centripetal acceleration due to the circular motion is

$$a_c = \frac{v^2}{R_e + h}. \tag{3}$$

Since gravity is providing the centripetal acceleration,

$$F = ma_c \implies \frac{GM_e m}{(R+h)^2} = \frac{v^2}{R+h}. \tag{4}$$

Solving for the velocity, we get,

$$v = \sqrt{\frac{GM_e}{(R_e + h)}}. \tag{5}$$

Thus, for circular orbit, the tangential velocity and radius of the orbit are related. If the height of the satellite is very small compared to the earths radius, then, we can neglect h in $R_e + h$ to get

$$v = \sqrt{\frac{GM_e}{R_e}} \tag{6}$$

for a satellite near the surface. Note that near the surface can really mean 100s of km above the surface, since the earths radius is HUGE!. Plugging in the numbers, this comes about to about 8km/s. That is the orbital speed of a satellite near the surface. Also, if it is very close to the surface, then

$$F \approx \frac{GM_e m}{R_e^2} = gm. \tag{7}$$

g of course is the acceleration due to gravity $(= \frac{GM_e}{R_e^2})$ near the earths surface that we are very familiar with! Plug in the numbers and see what you get for g. If we go far away then g changes.

Talking about going far away, it is considerable interest to understand how much work is done on raising an object by a certain amount in the presence of gravity. We all know that if we raise an object of mass m by a height h, then the amount of work we do is given by the gravitational potential energy, i.e., mgh. This however is only valid when we are near the earths surface. Let's do a general calculation of this work, and see how it reduces to this.

Consider an object of mass m at a distance r_1 from (the center of) another object of mass M. In order to move m to a radius of r_2 we have to do work, since M is trying to apply an attractive force. We have to apply a force thats opposite to it. If we move the object very slowly, so that its acceleration is almost zero, then the force we apply is equal and opposite to gravitational force. The work done by us is,

$$W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \frac{GMm}{r^2} dr = \left(-\frac{GMm}{r_2} \right) - \left(-\frac{GMm}{r_1} \right). \tag{8}$$

This looks like the change in the quantity in brackets. That quantity we call the gravitational potential energy, so that, the work done is equal to the change in the potential energy. Note the negative sign, and the $1/r$ dependence. The potential energy still increases as we go higher or farther away. Now, how does this reduce to mgh when we're near the earth. Suppose $r_2 = r_1 + h$ and $r_1 = R_e$. Lets also put $M = M_e$, the mass of the earth. Then

$$W = -GM_e m \left(\frac{1}{R_e + h} - \frac{1}{R_e} \right) = GMm \frac{h}{R_e(R_e + h)}. \tag{9}$$

If $h \ll R_e$, then we can neglect it in the denominator to get,

$$W = \frac{GMmh}{R_e^2} = mgh \tag{10}$$

from the definition of g above! So indeed, our old formula works when we're very close to the surface!

We will calculate one last thing - the initial velocity that we must impart to an object for it to leave earths gravitational pull. Note that the negative potential energy is indicative of the fact that the object is bound to the earth. Very far away the potential energy is zero. If we want an object to escape, then we must give it enough energy to "reach very far away", in other words, if we give it enough kinetic energy such that its total energy is zero, then very far away, when its velocity is close to zero, then energy conservation demands that its potential energy also be zero, thus making it escape from gravity. For an initial total energy of zero on the earths surface, we need,

$$\frac{1}{2}mv^2 - \frac{GM_e m}{R_e} = 0 \tag{11}$$

giving

$$v = \sqrt{\frac{2GM_e}{R_e}}. \tag{12}$$

This is approximately 11km/s, and at this initial speed launched from or near the earths surface, an object will escape.

Just as a visualization, we can think of all these different scenarios, i.e., projectile motion, orbital motion and an object escaping as essentially the same thing with increasing initial velocities. For small velocities, an object will fall back on the earth and execute projectile motion. For larger velocities, as the object starts falling back, the earth curves away from beneath it, and thus the object keeps falling and essentially is in orbit. For even higher velocities, the same thing happens, but the velocity is so high that falls to far and escapes earths gravitational effects.

3 Additional resources

- PhET demo on gravitational force and orbits - look at `http://phet.colorado.edu`